

Noncritical M-theory in $2 + 1$ dimensions as a nonrelativistic Fermi liquid

Petr Hořava and Cynthia A. Keeler

*Berkeley Center for Theoretical Physics and Department of Physics,
University of California,
Berkeley, CA 94720-7300, U.S.A., and
Theoretical Physics Group, Lawrence Berkeley National Laboratory,
Berkeley, CA 94720-8162, U.S.A.
E-mail: horava@berkeley.edu, ckeeler@berkeley.edu*

ABSTRACT: We claim that the dynamics of noncritical string theories in two dimensions is related to an underlying noncritical version of M-theory, which we define in terms of a double-scaled nonrelativistic Fermi liquid in $2 + 1$ dimensions. After reproducing Type 0A and 0B string theories as solutions, we study the natural M-theory vacuum. The vacuum energy of this solution can be evaluated exactly, its form suggesting a duality to the Debye model of phonons in a melting solid, and a possible topological nature of the theory. The physical spacetime is emergent in this theory, only for states that admit a hydrodynamic description. Among the solutions of the hydrodynamic equations of motion for the Fermi surface, we find families describing the decay of one two-dimensional string theory into another via an intermediate M-theory phase.

KEYWORDS: M-Theory, Superstrings and Heterotic Strings, Matrix Models, M(atr)ix Theories.

Contents

1. Introduction and summary	2
2. From noncritical strings to noncritical M-theory	4
2.1 Type 0A and 0B strings in two dimensions	4
2.2 Introducing noncritical M-theory	6
2.2.1 A parable on the relation between the radius and the string coupling	7
3. Nonperturbative M-theory as a double-scaled Fermi liquid	8
3.1 First thoughts on the double-scaling limit	9
3.2 Quantum mechanics of the double-scaled Fermi liquid	9
3.3 The moduli space of solutions	10
4. Examples of solutions I: Type 0A and 0B strings from M-theory	11
4.1 Type 0A	11
4.2 Type 0B	12
5. Examples of solutions II: the M-theory vacuum in 2 + 1 dimensions	13
5.1 Scaling at leading order in $1/N$	13
5.2 The double-scaling limit	16
5.3 The worldsheet cosmological constant and the string susceptibility	16
6. Exact vacuum energy in noncritical M-theory	17
6.1 The weak coupling expansion	18
6.1.1 Leading order	18
6.1.2 Higher loops	18
6.1.3 Summation of higher-genus Type 0A contributions	19
6.1.4 Summation of the leading logs	19
6.2 The strong coupling expansion	20
6.3 Dependence on the cutoff and volume	21
6.4 The exact formula	22
6.4.1 Evaluation by a contour integral	22
6.5 The exact formula in the weak coupling expansion	24
7. Analogy with the Debye model of phonons in solids	25
7.1 The Debye model	25
7.2 The analogy	26
7.3 Reintroducing the cutoff: the melting crystal interpretation	27
7.4 Solution with two Fermi surfaces as a universal melting crystal	28

8. Observables and symmetries	29
8.1 Observables	29
8.2 Symmetries	30
8.2.1 Massless modes vs. symmetries	30
9. Semiclassical spacetime physics as hydrodynamics of the Fermi liquid	31
9.1 Classical equations of motion for the Fermi surface	31
9.2 Vacua with q as the scaling variable	32
9.2.1 Duality to thermofield dynamics in the rightside-up harmonic potential	32
9.3 A family of stationary solutions	33
9.4 A twisted M-theory state	34
10. Time-dependent solutions	35
10.1 Losing or gaining a dimension	35
10.2 Solutions interpolating between two string vacua	36
11. Conclusions	37

1. Introduction and summary

In the wake of the second string revolution ten years ago, we have been left with a satisfying picture of a unique theory, with different string vacua connected by a web of dualities. It is somewhat ironic, however, that in the process of establishing that string theory is a unique theory, it was also discovered that this unique theory — provisionally called “M-theory” — is not always a theory of fundamental strings. Despite much progress in our understanding of M-theory in the last ten years, the nature of its degrees of freedom is still rather elusive, representing one of the major challenges of the field.

Ultimately, we wish to understand the landscape of all possible solutions of the theory. However, it is difficult to imagine how this would be possible in the absence of a clear understanding of the nature of the underlying degrees of freedom. On another note, it has long been suspected that the physical spacetime in quantum gravity should emerge as a derived concept. A more precise realization of this hope would also seem to require access to more fundamental degrees of freedom of quantum gravity.

In this paper, we will address these issues in the highly controlled (indeed, exactly solvable) context of noncritical string theories in two spacetime dimensions [1]–[8], as defined via their matrix model formulation [9]–[13]. We shall find that noncritical string theories are also connected in a larger framework, of a theory in $2 + 1$ dimensions which we refer to as “noncritical M-theory”. We give an exact, nonperturbative definition of this noncritical M-theory, from which many exact results can be obtained. In the process, we will get our first glimpse into the fundamental degrees of freedom in M-theory, at least in its $2 + 1$ -dimensional incarnation: Noncritical M-theory is a theory of double-scaled nonrelativistic

fermions in $2 + 1$ dimensions. This exact formulation of noncritical M-theory will allow us to understand in detail the entire space of solutions of the theory, the space frequently represented in full M-theory by the well-known “starfish” diagram.

The organization and outline of this paper are as follows. After a brief review of noncritical Type 0A and 0B strings in $1 + 1$ dimensions in section 2.1, we present our definition of noncritical M-theory in terms of a double-scaled Fermi liquid in $2 + 1$ dimensions in section 2.2. In particular, we propose to identify the extra dimension of M-theory with the angular dimension on the plane populated by the nonrelativistic fermions. The theory is further developed in section 3, where we also discuss the moduli space of all solutions of the theory, as well as the connection between the existence of hydrodynamic degrees of freedom and the existence of a semiclassical spacetime description of a given solution. In section 4, we reproduce the linear dilaton vacua of two-dimensional Type 0A and 0B noncritical string theories as solutions of noncritical M-theory. In section 5, we introduce the natural M-theory vacuum. First we analyze the scaling at the leading order in large N and identify the natural scaling variable μ , and then define the nonperturbative double-scaling limit of this vacuum.

Section 6 contains some of the central results of this paper. In particular, we present an exact calculation of the vacuum energy of the M-theory vacuum solution, as a function of the scaling variable μ . The exact formula for the vacuum energy turns out to be one-loop exact (in perturbation theory in the powers of $1/\mu \sim \kappa^{2/3}$), with an infinite series of instanton-like corrections, each of which is also one-loop exact. This result is suggestive of a possible topological nature, or at least localization of the path integral, of noncritical M-theory. In section 7 we point out that the exact formula for the vacuum energy suggests a dual interpretation, in terms of the Debye model of a quantum crystal at finite temperature set by the string scale. In fact, μ controls how many atoms have been *removed* from a large Debye crystal, leading to an interpretation in terms of crystal melting.

In section 8 we address two more general aspects of noncritical M-theory: Its observables and symmetries. A particularly natural observable is given by the density of eigenvalues. This observable is the M-theory analog of the massless tachyon from noncritical string theories. The theory is shown to exhibit an infinite \mathcal{W} symmetry algebra. Section 9 develops a general framework for identifying “good” hydrodynamic solutions of the theory, for which a spacetime description should be possible. We formulate the classical hydrodynamic equation of motion for the Fermi surface, and present several simple static solutions of this equation. A surprising duality to the thermofield dynamics of fermions in the rightside-up harmonic oscillator potential is found. Section 10 continues the analysis by introducing a general class of time-dependent solutions of the Fermi surface equations of motion. Among the time-dependent solutions, we find classes representing a dynamical change of the spacetime dimension. In particular, there are solutions describing the decay of a $1 + 1$ -dimensional string theory vacuum to another one via an intermediate $2 + 1$ -dimensional M-theory phase. Section 11 concludes with some general remarks and some open questions.

2. From noncritical strings to noncritical M-theory

Our starting point is the matrix model formulation of various noncritical strings in two spacetime dimensions. We concentrate on the Type 0A and 0B superstrings [1, 2], but our analysis can be easily extended to include other vacua, such as the Type II or bosonic strings in two dimensions.

2.1 Type 0A and 0B strings in two dimensions

This is not the right place for a lengthy overview of two-dimensional strings, and we only highlight some basic aspects as needed for the rest of the paper. Excellent extensive reviews of the subject exist, see, e.g., [3]–[8].

Type 0 superstrings are defined via a double-scaling limit of the Euclidean matrix path integral

$$\mathcal{Z} = \int DM(t) e^{-S(M)}. \tag{2.1}$$

In Type 0B theory [2, 1], the action is given by

$$S_{0B}(M) = \beta N \int dt \text{Tr} \left(\frac{1}{2} (D_t M)^2 + V(M) \right). \tag{2.2}$$

M is a Hermitian $N \times N$ matrix, D_t is the covariant derivative with respect to a $U(N)$ gauge field A_0 , and β is a coupling constant which can be conveniently reabsorbed into M .

The Type 0A superstring similarly corresponds to a quiver matrix mechanics [1],

$$S_{0A}(M, M^\dagger) = \beta N \int dt \left(\text{Tr} [(D_t M)^\dagger D_t M + V(M, M^\dagger)] \right). \tag{2.3}$$

In this case, M is an $N \times (N+q)$ complex matrix, and the gauge group is $U(N) \times U(N+q)$. q is interpreted as the net D0-brane charge or, alternatively, the value of the RR two-form flux in the vacuum. M is the matrix of open-string tachyon modes on the system of $N+q$ D0-branes and N anti D0-branes in Type 0A theory, or N unstable D0-branes in the Type 0B matrix model, along the lines of [9].

The universal part of the potential is

$$V(M) = -\frac{1}{2} \omega_0^2 M^2 + \dots \tag{2.4}$$

Here the “...” stand for stabilizing, nonuniversal terms in the potential, and ω_0 is the fundamental frequency scale of the theory. In Type 0A and 0B string theories, this fundamental frequency sets the string scale, $\omega_0 = 1/\sqrt{2\alpha'}$.

In the singlet sector, the matrix models reduce to a theory of N free fermions, representing the locations of N eigenvalues y_α , $\alpha = 1, \dots, N$ of M along a spatial dimension y . The ground state of this system corresponds to all states filled up to a (negative) Fermi energy ε_F . The second-quantized Hamiltonian is

$$\mathcal{H} = \beta N \int dy \left(-\frac{1}{2(\beta N)^2} \partial_y \psi^\dagger \partial_y \psi + V(y) \psi^\dagger \psi \right). \tag{2.5}$$

Clearly, the role of the Planck constant is played by $\hbar \equiv 1/(\beta N)$.

The double-scaling limit of the system corresponds to taking the $N \rightarrow \infty$ limit with $\varepsilon_F \rightarrow 0$ while keeping $\mu \equiv -N\varepsilon_F$ fixed. It is convenient to introduce the rescaled spatial dimension λ ,

$$\lambda = \sqrt{\beta N} y. \tag{2.6}$$

After the double-scaling limit, the single-particle equation becomes

$$\left(-\frac{1}{2} \frac{\partial^2}{\partial \lambda^2} + V(\lambda) \right) \psi(\lambda) = \nu \psi(\lambda), \tag{2.7}$$

where ν is the double-scaled energy eigenvalue, and

$$V(\lambda) = \begin{cases} -\frac{1}{2} \omega_0^2 \lambda^2 & \text{for Type 0B,} \\ -\frac{1}{2} \omega_0^2 \lambda^2 + \frac{(g^2 - \frac{1}{4})}{\lambda^2} & \text{for Type 0A.} \end{cases} \tag{2.8}$$

The careful definition of the double-scaling limit involves introducing a nonuniversal stabilizing regulator Λ , which we will represent by cutting off the potential by an infinite wall at $y \sim 1$. In the double-scaled variable λ , this amounts to placing an infinite wall at $\lambda = \sqrt{2\Lambda} \sim \sqrt{N}$.¹

Since \hbar is proportional to $1/N$, the large N limit that we are interested in corresponds to the semiclassical limit of the system. In the WKB approximation, the semiclassical fermions occupy a certain area in phase space, and we have

$$N = \int \frac{dp d\lambda}{2\pi \hbar} \theta \left(\varepsilon_F - \frac{p^2}{2} - V(\lambda) \right). \tag{2.9}$$

In a given static vacuum state, one of the main quantities of interest to calculate is the vacuum energy

$$F = \lim_{\mathcal{T} \rightarrow \infty} \left(-\frac{1}{\mathcal{T}} \log \mathcal{Z} \right), \tag{2.10}$$

with \mathcal{T} is the total length of the Euclidean time dimension. In the limit of $\mathcal{T} \rightarrow \infty$, this is reduced to the evaluation of the energy of the ground state,

$$F = \frac{E_0}{\hbar} = \frac{1}{\hbar} \sum_{k=1}^N \nu_k, \tag{2.11}$$

the sum being performed up to the Fermi energy $N\varepsilon_F \equiv \nu_N$. In the double-scaling limit, F represents (a nonperturbative completion of) the string partition function, and can be expanded to match the perturbative sum over all worldsheet topologies, i.e., over all genera of connected Riemann surfaces. It can be exactly evaluated by first defining the density of states $\rho(\mu)$,

$$\rho(\mu) = \hbar \sum \delta(-\mu - \nu_n), \tag{2.12}$$

¹For a clear discussion of the technical details of the double-scaling limit, see, e.g., [14].

and observing that in terms of $\rho(\mu)$, we have

$$\frac{\partial F}{\partial \Delta} = \frac{1}{\pi} \mu, \quad \frac{\partial \Delta}{\partial \mu} = \pi \rho(\mu). \tag{2.13}$$

Here Δ is another scaling variable, usually referred to in the matrix models of noncritical strings as the “worldsheet cosmological constant.” The logarithmic scaling $\rho(\mu) \sim \log \mu$ is a signature behavior of two-dimensional string theory [10]. With the use of (2.13), this behavior implies for the expansion of F in the powers of the string coupling $g_s \sim 1/\mu$

$$F(\mu) \sim \mu^2 \ln \mu + \ln \mu + \mathcal{O}(1/\mu^2). \tag{2.14}$$

The log terms come from the leading $\log \mu$ behavior of the density of states, and are characteristic of noncritical string theory in two dimensions (in the linear dilaton background, screened by the Liouville wall). The string coupling is determined via $\mu \sim g_s^{-1}$, and the two terms have a clear interpretation: While the first one is the tree-level contribution from worldsheets of spherical topology, the second term is a one-loop contribution from the torus. The $\log \mu$ term — or, more exactly, $\log(\Lambda/\mu)$ with Λ the cutoff — is properly interpreted as the volume of the Liouville dimension.

The theory is nonperturbatively fully defined via its free-fermion formulation. A non-local transform maps the eigenvalue coordinate to the physical spacetime, in which the systems can be understood in terms of a spacetime effective theory of strings. However, this transformation only exists under special circumstances, when the N fermions are distributed such that the quantum state of the Fermi system can be bosonized in terms of hydrodynamic degrees of freedom, such as the fluctuations of the Fermi surface. These fluctuations then correspond in the physical spacetime picture to the massless tachyon (and, in Type 0B, the RR scalar) of noncritical string theory.

2.2 Introducing noncritical M-theory

The spectrum of noncritical Type 0A string theory contains stable D0-branes, which couple to a RR one-form gauge field. It admits vacua with a nonzero value of the RR flux q . This flux can also be interpreted as the net number of D0-branes sustaining the background. In the matrix model, q is represented as the difference between the number of rows and columns of M .

In the critical Type IIA superstring, stable D0-branes are interpreted as KK momentum modes along a hidden, eleventh dimension of M-theory. It is natural to ask whether a similar interpretation can be found for the stable D0-branes of the noncritical Type 0A theory, perhaps leading to a noncritical version of M-theory in $2+1$ dimensions. This question can be addressed from several points of view. For example, one can try to identify the lift of the effective spacetime action of Type 0A theory to an effective theory in $2+1$ dimensions. Alternatively, one can search for an implementation of the lift to M-theory directly in the matrix model. In this paper, we will circumvent some apparent difficulties with these two approaches, by addressing the question directly in the language of the second-quantized double-scaled fermions.

It has been observed in [1] that the eigenvalue coordinate λ of the Type 0A matrix model can be thought of as the radial coordinate on a two-dimensional plane (which we will refer to as the “eigenvalue plane” from now on). From this viewpoint, the Type 0A vacuum at fixed RR flux q can be interpreted as the sector with fixed angular momentum $J = q$ in a $2 + 1$ dimensional theory of fermions on the eigenvalue plane. Since one unit of the D0-brane charge corresponds to one unit of the angular momentum, this leads us to a natural lift of the Type 0A vacua to M-theory:

We propose to identify the extra dimension of noncritical M-theory with the angular variable on the eigenvalue plane of the double-scaled nonrelativistic Fermi system in the upside-down harmonic potential.

The remainder of this paper can be viewed as a series of tests justifying this definition of noncritical M-theory and its proposed relation to the dynamics of noncritical strings.

2.2.1 A parable on the relation between the radius and the string coupling

At first, the proposed identification of the third dimension of M-theory with the angular dimension on the eigenvalue plane may seem somewhat counterintuitive. It suggests that the weakly coupled region in Type 0A string theory is associated with the region where the radius of the angular S^1 dimension of noncritical M-theory is large; similarly, the strongly coupled regime of string theory corresponds to the region near the origin on the eigenvalue plane where the radius of the angular S^1 is small. In contrast, critical M-theory in eleven dimensions relates the strong string coupling regime to the large extra dimension of M-theory.

In order to illustrate that the intuition based on eleven-dimensional M-theory may be incorrect in low enough dimensions, consider the following parable, which begins with the Einstein-Hilbert action in D spacetime dimensions X^μ ,

$$S = \frac{1}{\mathcal{G}_D} \int d^D X \sqrt{G} \mathcal{R}(G), \tag{2.15}$$

with $G_{\mu\nu}$ the spacetime metric and \mathcal{G}_D the Newton constant. We compactify to $D - 1$ dimensions on S^1 , parametrized by coordinates $(X^\mu) = (x^i, Y)$, $i = 1, \dots, D - 1$, with $Y = Y + 2\pi$. The metric can be decomposed as

$$G_{\mu\nu} dX^\mu dX^\nu = e^{2a\Phi} g_{ij} dx^i dx^j + e^{2b\Phi} dY^2, \tag{2.16}$$

where Φ is a scalar field (to be identified with the string theory dilaton), g_{ij} is the (string frame) metric in $D - 1$ spacetime dimensions, and a and b are constants to be determined below. We shall only keep the zero modes of all fields on S^1 , and for simplicity also drop the off-diagonal, Abelian gauge field part of $G_{\mu\nu}$. Using this decomposition (2.16), the Einstein-Hilbert action (2.15) becomes

$$S = \frac{2\pi}{\mathcal{G}_D} \int d^{D-1} x \sqrt{g} \left(e^{[(D-3)a+b]\Phi} R(g) + \dots \right), \tag{2.17}$$

where “...” refer to terms that depend on the derivatives of Φ , and $R(g)$ is the scalar curvature of the lower-dimensional metric g_{ij} .

If (2.17) is to be the leading term of the effective string-theory action in the string frame, with Φ the conventionally normalized dilaton (i.e., $e^\Phi = g_s$), the power of e^Φ in (2.17) must equal -2 , implying

$$(D - 3)a + b = -2, \quad g_s = e^\Phi, \quad R_D = e^{b\Phi}, \quad (2.18)$$

where the third relation — between the radius R_D of the extra dimension measured in the D -dimensional Planck units and the dilaton — follows from (2.16). When $D = 3$, the first equation in (2.18) implies that $b = -2$, independently of the value of a . Generally, one more relation is needed to determine the value of a ; this extra relation could for example come from the requirement that the kinetic term of Φ be correctly normalized, or from a different constraint. In any case, in $D = 3$ we do not need to know a to make our point: Since $b = -2$ in $D = 3$, the second and third relation in (2.18) imply that the size of the third dimension, measured in the three-dimensional Planck units, comes out inversely proportional to the square of the string coupling,

$$R_3 \sim \frac{\mathcal{G}_3}{g_s^2}. \quad (2.19)$$

Thus, we see that in the reduction of the simple Einstein-Hilbert Lagrangian from three to two dimensions, the large radius of the extra dimension of M-theory corresponds to the *weak* string coupling constant, while the strong string coupling regime is described by the *small* radius of the M-theory dimension. This may be counterintuitive from the viewpoint of the critical M-theory in eleven dimensions, but seems compatible with the possibility of interpreting the third dimension of noncritical M-theory as the angular dimension on a plane.

Of course, our simple parable has at least two caveats: First of all, the eigenvalue plane should not be directly identified with the physical spacetime. Instead, they should be related by a nonlocal transform analogous to the transform between the eigenvalue dimension and the Liouville dimension in noncritical string theory. Secondly, the full effective action of noncritical M-theory in the physical three-dimensional spacetime is likely to be much more complicated than the simple Einstein-Hilbert Lagrangian considered in the parable.

3. Nonperturbative M-theory as a double-scaled Fermi liquid

Now we can systematically develop the theory from first principles, and check that it leads to sensible results.

We start with a nonrelativistic spinless Fermi field $\hat{\Psi}(t, y_1, y_2)$ in $2 + 1$ dimensions, before double scaling. In the double scaling limit, $\hat{\Psi}$ turns into a double-scaled Fermi field $\Psi(t, \lambda_1, \lambda_2)$, described by the action

$$S_M = \int dt d^2\lambda \left(i\Psi^\dagger \frac{\partial \Psi}{\partial t} - \frac{1}{2} \sum_{i=1,2} \frac{\partial \Psi^\dagger}{\partial \lambda_i} \frac{\partial \Psi}{\partial \lambda_i} + \frac{1}{2} \omega_0^2 \sum_{i=1,2} \lambda_i^2 \Psi^\dagger \Psi + \dots \right). \quad (3.1)$$

Here the “...” stand for nonuniversal regulating and stabilizing terms in the potential. We will represent them by an infinite wall placed at $\lambda = \sqrt{2\Lambda}/\omega_0$. In the units where \hbar is

dimensionless, the basic variables ω_0 , t , λ_i , and the momentum p_i conjugate to λ_i have dimensions 1, -1 , $-1/2$ and $1/2$, respectively. Until further notice, we will Wick rotate t and interpret it as the Euclidean time coordinate.

3.1 First thoughts on the double-scaling limit

The double-scaling limit has two ingredients, which are not always clearly separated in the studies of two-dimensional string theory. Both steps are performed simultaneously, but the first step is more universal while the second one is specific to a given solution.

- (1) Eliminate the nonuniversal features of the potential, represented by the cutoff dependence, and take the large- N limit;
- (2) Choose a state, i.e., a distribution of N fermions among the available states, whose double-scaling limit is taken. Identify the scaling variable to be held fixed as $N \rightarrow \infty$. Typically, the scaling variable is a combination of N and a conserved quantity such as the energy of the Fermi surface or its angular momentum.²

Some simple modifications of this process can be easily implemented, one example being the situation when we do not hold the number of fermions N fixed, but instead fix a chemical potential. We will not distinguish such modifications from our prescription.

3.2 Quantum mechanics of the double-scaled Fermi liquid

The theory can be easily quantized. There are two useful representations. In the first one, we use the Cartesian coordinates λ_i , and view the system as two decoupled upside-down harmonic oscillators. In this representation, the second-quantized Fermi field Ψ can be expanded in terms of products of Type 0B wavefunctions as follows,

$$\Psi(t, \lambda_i) = \int d^2\nu \sum_{s_1, s_2 = \pm} a_{s_1 s_2}(\nu_1, \nu_2) \psi_{s_1}(\nu_1, \lambda_1) \psi_{s_2}(\nu_2, \lambda_2) e^{-i(\nu_1 + \nu_2)t}, \quad (3.2)$$

where ν_i are the energy levels of the two one-dimensional upside-down oscillators, and $s_i = \pm$ are the parity quantum numbers of the Type 0B wavefunctions. The annihilation operators $a_{s_1 s_2}(\nu_1, \nu_2)$ and their conjugates satisfy the canonical anticommutation relations,

$$\{a_{s_1 s_2}(\nu_1, \nu_2), a_{s'_1 s'_2}^\dagger(\nu'_1, \nu'_2)\} = \delta_{s_1 s'_1} \delta_{s_2 s'_2} \delta^2(\nu_i - \nu'_i). \quad (3.3)$$

Alternatively, we can use a representation in terms of polar coordinates λ, θ on the eigenvalue plane, expanding Ψ in a complete basis of Type 0A wavefunctions

$$\Psi(t, \lambda_i) = \sum_{q \in \mathbf{Z}} e^{iq\theta} \int d\nu a_q(\nu) \psi_q(\nu, \lambda) e^{-i\nu t}, \quad (3.4)$$

supplemented with the canonical commutation relations

$$\{a_q(\nu), a_{q'}^\dagger(\nu')\} = \delta_{qq'} \delta(\nu - \nu'). \quad (3.5)$$

²We define the Fermi surface more generally as the boundary between the filled and empty regions in phase space.

In these formulas, q is the value of the Type 0A RR flux, interpreted in the M-theory context as the angular momentum on the eigenvalue plane. The Type 0A and 0B wavefunctions $\psi_{s_1 s_2}(\nu_1, \nu_2)$ and $\psi_q(\nu)$ are given explicitly in terms of cylindric Whittaker functions [3, 15, 18].

3.3 The moduli space of solutions

The simplicity of the quantum mechanics of the double-scaled Fermi system allows us to make some general remarks about the space of all solutions of noncritical M-theory. These observations will be illustrated in specific examples in the rest of the paper.

In the double scaling limit, the nonuniversal anharmonic pieces in the potential are scaled away, and the double-scaled Fermi theory becomes free. This leads to a particularly simple description of all possible quantum states in this theory. In order to specify a quantum state $|\text{phys}\rangle$, we simply need to decide how each canonical pair of oscillators a, a^\dagger acts on $|\text{phys}\rangle$. Any quantum state that can be prepared by the infinite collection of fermionic oscillators is a solution of noncritical M-theory. Most such states will not have a clear semiclassical description in terms of collective bosonic degrees of freedom, since generally each canonical pair can act on $|\text{phys}\rangle$ in a way uncorrelated with the action of the other pairs. Only those states for which the fermionic oscillators act in a highly correlated way will exhibit semiclassical hydrodynamic bosonic excitations. We will refer to such states generically as “hydrodynamic states.” We expect that the excitations of such a hydrodynamic state can be described in terms of an effective action for the fluctuations of the hydrodynamic bosonic variables (such as the bosonic fluctuations of the Fermi surface, whenever the latter can be defined). Only states that can be so bosonized can be described in terms of low-energy quantum gravity in a semiclassical spacetime. The physical spacetime itself is an emergent property of the hydrodynamic states, and is related in a complicated nonlocal way to the eigenvalue plane on which the fermions reside.

It is worth noting that our definition of noncritical M-theory in terms of free fermions leads to a very precise refinement of the famous “starfish diagram,” traditionally drawn to illustrate the space of all vacua in critical string/M theory. This starfish diagram usually depicts several asymptotic corners, in which perturbative string/M-theory descriptions are available, connected into a single moduli space whose middle portion remains rather mysterious. In contrast, as we have just argued, the problem of identifying the space of all solutions in our noncritical M-theory is effectively reduced to a simple, mathematically well-posed problem, essentially equivalent to the representation theory of the algebra of the infinite set of decoupled canonical fermionic oscillators $a_q^\dagger(\nu)$ and $a_q(\nu)$.

In this picture, the spacetime effective field theory description is effectively equivalent to the hydrodynamics of the Fermi liquid. Whether or not an effective bosonization to a spacetime description exists, however, the physics of any given solution is always nonperturbatively fully defined by the underlying fundamental degrees of freedom of noncritical M-theory, the double-scaled nonrelativistic free fermions on the eigenvalue plane. Different quantum vacua of the system correspond to different separable Hilbert spaces that can be built as fermionic Fock spaces from a given ground state. This leads to an intricate picture of a web of Hilbert spaces, representing all possible ways in which the N fermions can

occupy the available single-particle states while the double-scaling limit is taken. Some such states represent static vacuum solutions, others will describe excited states in such vacua (i.e., they belong to the Hilbert space for which the corresponding vacuum state is the ground state). Some solutions will be time dependent, interpolating between different static vacua at early and late times. Yet others may represent big-bang/big-crunch cosmologies, evolving from/to M-theory states with no conventional semiclassical spacetime interpretation. Some may have a 2 + 1-dimensional spacetime, some reduce to string vacua in a 1 + 1-dimensional spacetime. Some solutions will have a dynamically changing spacetime dimension, evolving for example from 1 + 1 at early times via 2 + 1 at intermediate times to 1 + 1 at late times, etc. Some simple examples of such classes of solutions will be discussed below, but many more can be identified and studied within this rich and mathematically well-defined “landscape of all vacua” of noncritical M-theory.

4. Examples of solutions I: Type 0A and 0B strings from M-theory

As a first check that our definition of noncritical M-theory is acceptable, we shall reproduce known Type 0A and 0B vacua as its solutions.

4.1 Type 0A

Using the polar-coordinate representation of the theory, we first choose a value of the RR flux q , and define the Type 0A state $|0A, q, \mu\rangle$ as a solution of noncritical M-theory, as follows. The N fermions are distributed such that the Fermi sea is filled up to some (negative) Fermi energy $-\mu$ in the sector with angular momentum q while keeping the Fermi sea empty in all the sectors with angular momenta $q' \neq q$:

$$\begin{aligned} a_q(\nu) |0A, q, \mu\rangle &= 0 && \text{for } \nu > -\mu, \\ a_q^\dagger(\nu) |0A, q, \mu\rangle &= 0 && \text{for } \nu < -\mu, \\ a_{q'}(\nu) |0A, q, \mu\rangle &= 0 && \text{for all } \nu \text{ with } q' \neq q. \end{aligned} \tag{4.1}$$

Notice that it is important to use this definition while taking the double scaling limit. In particular, this state is not equivalent to sending $\mu \rightarrow \infty$ in all sectors with $q' \neq q$ after the double scaling limit has been performed. To see this, we shall now reproduce the known result for the exact vacuum energy of the Type 0A solution, from a direct M-theory calculation.

The total vacuum energy of the 0A state $|0A, q, \mu\rangle$ will be equal to the sum of vacuum energies over all M-theory sectors of fixed angular momentum q , filled up to a q -dependent Fermi level as indicated in (4.1). The naive limit $\mu \rightarrow \infty$ in sectors of $q' \neq q$ is properly interpreted as a prescription to keep the Fermi level at the cutoff Λ during the double scaling limit. Recall that at fixed q , the density of states in Type 0A theory has an asymptotic string-coupling expansion [1],[16]–[19]³

$$\rho_{0A}(\mu, q) \approx -\frac{1}{4\pi} \log(\mu^2 + q^2) + \mathcal{O}(1/\mu^2). \tag{4.2}$$

³Throughout the paper, we use the “ \approx ” symbol to denote exact asymptotic expansions, reserving “ \sim ” to represent a more loosely defined proportionality or scaling relation.

Sending formally $\mu \rightarrow \infty$ would kill all the terms $\mathcal{O}(1/\mu)$ and higher, but we would still be left with the leading log. Keeping track of the cutoff dependence in ρ_{0A} during the double scaling, we find an extra correction $\sim \frac{1}{2\pi} \log \Lambda$ in the density of states (see, e.g., [14]). Thus, setting $\mu = \Lambda$ in all sectors $q' \neq q$ and taking the double scaling limit $\Lambda \rightarrow \infty$ will eliminate also the log contribution from all sectors of $q' \neq q$, and the resulting density of states of this M-theory solution is manifestly equal to the density of states in Type 0A theory at RR flux q . The integration of ρ to obtain the vacuum energy is then straightforward.

4.2 Type 0B

The Type 0B linear dilaton vacuum $|0B, \mu\rangle$ can be defined as a solution of noncritical M-theory as follows. In the Cartesian representation of the theory, the energies ν_1 and ν_2 of the two one-dimensional oscillators are separately conserved. Fix all the quantum numbers of the second oscillator, i.e. pick an arbitrary fixed value $\bar{\nu}_2$ of ν_2 and \bar{s}_2 of s_2 , and fill all states in the sector with $\nu_2 = \bar{\nu}_2$ and $s_2 = \bar{s}_2$ up to a negative Fermi energy $-\mu$ while keeping the Fermi sea empty in all sectors with $\nu_2 \neq \bar{\nu}_2$ or $s_2 \neq \bar{s}_2$:

$$\begin{aligned}
 a_{s_1 s_2}(\nu_1, \nu_2) |0B, \mu\rangle &= 0 && \text{for } \nu_1 > -\mu \text{ with } \nu_2 = \bar{\nu}_2 \text{ and } s_2 = \bar{s}_2, \\
 a_{s_1 s_2}^\dagger(\nu_1, \nu_2) |0B, \mu\rangle &= 0 && \text{for } \nu_1 < -\mu \text{ with } \nu_2 = \bar{\nu}_2 \text{ and } s_2 = \bar{s}_2, \\
 a_{s_1 s_2}(\nu_1, \nu_2) |0B, \mu\rangle &= 0 && \text{for all } \nu_1 \text{ with } \nu_2 \neq \bar{\nu}_2 \text{ or } s_2 \neq \bar{s}_2.
 \end{aligned}
 \tag{4.3}$$

Several observations:

- Unlike in Type 0A, selecting a fixed value $\nu_2 = \bar{\nu}_2$ to fill the Fermi sea does not introduce any new physical free parameter, as any change in the value of $\bar{\nu}_2$ can be absorbed in a shift of μ . Without any loss of generality, we can take $\bar{\nu}_2 = 0$.
- The parallels between the Type 0A and Type 0B constructions are even stronger before the double-scaling limit. In that situation, ν_2 is also a discrete conserved quantum number.
- The bosonic $c = 1$ string can also be easily found as a solution of noncritical M-theory, by repeating the steps of our Type 0B construction and filling only one side of the one-dimensional effective potential at fixed $\nu_2 = \bar{\nu}_2$ and $s_2 = \bar{s}_2$.
- Similarly, our construction can be easily extended to simple orbifolds of Type 0A and 0B theories, such as the IIA and IIB models considered in [20–22].⁴
- The quantum states defining the Type 0A and 0B theories exhibit a semiclassical Fermi surface which is effectively of higher codimension in phase space, compared to the naive ground state of the system (to which we return in section 5). This is

⁴We also mention in passing that a duality diagram has been proposed some time ago for critical 0A and 0B in ten dimensions in [23], conjecturally connecting them to nonsupersymmetric compactifications of M-theory. Our results do not have any direct bearing on whether or not the proposal of [23] is correct. Unlike Type 0 theories in the critical dimension, the two-dimensional models that we consider do not suffer from instabilities, and the duality properties are thus under control, and amenable to our exact analysis.

somewhat reminiscent of the higher-codimension Fermi surfaces classified and related to K-theory in [24], the main difference being that the system of spinless fermions is not in the stable regime of K-theory. Defining the proper semiclassical limit of such states at large N might require the more systematic approach to large N developed in [25].

5. Examples of solutions II: the M-theory vacuum in 2 + 1 dimensions

In critical M-theory, perhaps the most interesting vacua are those that exhibit the largest spacetime symmetry in uncompactified eleven dimensions: The flat Minkowski space, described at low energies by eleven-dimensional supergravity, and the heterotic M-theory solution [26], with the additional E_8 super Yang-Mills at the boundary of spacetime. It is in those solutions where the non-stringy character of M-theory is most prominent, since neither of these two vacua admits string-like excitations. Having reproduced the two-dimensional string theory vacua from our noncritical M-theory in the previous subsection, we can now analyze its “non-stringy phase,” and in particular, its 2 + 1 dimensional vacua.

The noncritical M-theory has one particularly natural solution, corresponding to filling the states up to some common Fermi energy ε_F in the 2+1 dimensional system of fermions, irrespective of their other quantum numbers. We will refer to this solution as the “M-theory vacuum.” By construction, it represents the M-theory lift of the linear dilaton vacua of Type 0A and 0B theories. Thus, we define the M-theory vacuum state $|M, \mu\rangle$ — using, for definiteness, the polar-coordinate representation of the theory — as follows:

$$\begin{aligned} a_q(\nu) |M, \mu\rangle &= 0 && \text{for } \nu > -\mu \text{ and all } q, \\ a_q^\dagger(\nu) |M, \mu\rangle &= 0 && \text{for } \nu < -\mu \text{ and all } q. \end{aligned} \tag{5.1}$$

Strictly speaking, one should distinguish between the definition of the M-theory state before and after the double scaling limit. However, we shall keep the distinction implicit, in order to keep the notation simple. We shall now analyze the scaling properties of this state, in order to identify appropriately the double-scaling limit of $|M, \mu\rangle$.

5.1 Scaling at leading order in $1/N$

The large N limit corresponds to the WKB approximation of the M-theory vacuum defined in (5.1). In this limit, the semiclassical density of states is given by

$$\rho(\nu) = \hbar \int \frac{d^2 p d^2 y}{(2\pi\hbar)^2} \delta(\nu - h(p_i, y_i)), \tag{5.2}$$

where the single-particle Hamiltonian is

$$h(p_i, y_i) = \frac{1}{2} \sum_{i=1,2} (p_i^2 - \omega_0^2 y_i^2 + \dots), \tag{5.3}$$

where we will only keep track of the universal part in the potential. Introducing $y = \sqrt{y_1^2 + y_2^2}$ and $p = \sqrt{p_1^2 + p_2^2}$, and switching to the polar coordinates separately in the

coordinate and momentum space, gives

$$\rho(\nu) = \int \frac{dp dy}{\hbar} p y \delta\left(\nu - \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 y^2\right). \quad (5.4)$$

We will use a rotationally invariant cutoff Λ , equivalent to placing an infinite wall at $\lambda \leq \sqrt{2\Lambda}/\omega_0$. The integration gives

$$\rho(\nu) = \frac{1}{\hbar} \int_{\sqrt{-2\varepsilon_F/\omega_0}}^{\sqrt{2\Lambda/\omega_0}} y dy \sim \frac{1}{\hbar\omega_0^2}(\varepsilon_F + \Lambda) \sim \frac{\varepsilon_F}{\hbar\omega_0^2}, \quad (5.5)$$

where in the final step we have dropped the nonuniversal cutoff-dependent part of the density of states, keeping only its dependence on ε_F .

We can use this evaluation of the density of states to obtain an expression for the vacuum energy of the system. In the semiclassical regime, each fermion occupies a unit volume $1/(2\pi\hbar)^2$ in phase space, and the total number N of fermions measures the semiclassical area of the filled region,

$$N(\varepsilon_F) = \int \frac{d^2p d^2y}{(2\pi\hbar)^2} \theta(\varepsilon_F - h(p_i, y_i)). \quad (5.6)$$

Similarly, the semiclassical ground-state energy is given by

$$E_0(\varepsilon_F) = \int \frac{d^2p d^2y}{(2\pi\hbar)^2} h(p_i, y_i) \theta(\varepsilon_F - h(p_i, y_i)). \quad (5.7)$$

Taking the derivative of each of those equations with respect to ε_F , we get

$$\frac{\partial N(\nu)}{\partial \nu} = \int \frac{d^2p d^2y}{(2\pi\hbar)^2} \delta(\nu - h(p_i, y_i)) \quad (5.8)$$

and

$$\frac{\partial E_0(\nu)}{\partial \nu} = \int \frac{d^2p d^2y}{(2\pi\hbar)^2} h(p_i, y_i) \delta(\nu - h(p_i, y_i)) = \nu \frac{\partial N}{\partial \nu}. \quad (5.9)$$

Since the density of states is related to N via

$$\rho(\nu) = \hbar \frac{\partial N}{\partial \nu}, \quad (5.10)$$

we finally obtain

$$F(\varepsilon_F) \equiv \frac{E_0(\varepsilon_F)}{\hbar} = \frac{1}{\hbar} \int^{\varepsilon_F} d\nu \nu \frac{\partial N}{\partial \nu} = \frac{1}{\hbar^2} \int^{\varepsilon_F} d\nu \nu \rho(\nu) \sim \frac{1}{\hbar^3 \omega_0^2} \left(\frac{\varepsilon_F^3}{3} + \frac{\varepsilon_F^2 \Lambda}{2} \right) \sim \frac{\varepsilon_F^3}{3\hbar^3 \omega_0^2}. \quad (5.11)$$

In the final step, we have again kept only the universal dependence on ε_F .⁵

From this central result, we can draw several interesting lessons:

⁵From now on, we set $\beta = 1$ by rescaling the corresponding variables. Hence, $\hbar = 1/N$.

(1) The natural scaling variable suggested by this lowest-order result is

$$\mu = -\varepsilon_F/\hbar \equiv -N\varepsilon_F. \tag{5.12}$$

It is satisfying to find that the scaling variable in M-theory is indeed the same as in Type 0A and 0B theory.

(2) The vacuum energy $F \sim \int \nu \rho(\nu) d\nu$ scales as

$$F \sim -\mu^3 + \dots \tag{5.13}$$

This behavior seems characteristic of M-theory. In the physical spacetime interpretation of this result, the leading term in F should correspond to the tree-level contribution, proportional to κ^{-2} where by κ we denote the spacetime coupling constant, possibly related to the Newton constant. This implies that in terms of κ , the natural expansion parameter $1/\mu$ of noncritical M-theory is

$$\frac{1}{\mu} = \kappa^{2/3}. \tag{5.14}$$

This, of course, is the behavior observed in critical heterotic M-theory [26]. It is also suggestive of a possible existence of membranes in the noncritical M-theory.

(3) Viewed in the context of large N theories, our $2 + 1$ dimensional model exhibits an interesting behavior: At the leading order at large N , our vacuum energy scales as

$$F \sim N^3, \tag{5.15}$$

to be contrasted with the more conventional $F \sim N^2$ behavior familiar from the traditional “planar” large N limit.

(4) Unlike in string theory in $1 + 1$ dimensions, there is no logarithmic dependence of the density of states on the Fermi energy. In noncritical strings, such a logarithmic dependence signifies the volume dependence of various terms in the sum over surfaces; its absence here suggests that the dependence on volume is reduced in M-theory. We shall return to this point in section 6.3, where the volume/cutoff dependence of the M-theory amplitudes will be discussed.

(5) As in noncritical string theory in $1 + 1$ dimensions, the system exhibits a particle-hole duality, accompanied by the exchange⁶

$$\mu \rightarrow -\mu. \tag{5.16}$$

Notice that the Fermi surface undergoes a topology changing transition as μ goes from positive to negative values, but the geometry of the Fermi surface after the transition

⁶This is an important symmetry, since it is related in the string theory context to the orbifold that produces Type IIA and IIB out of Type 0A and 0B vacua [20–22]. It is satisfying to see that a similar symmetry, and hence an orbifold procedure, extends to noncritical M-theory.

is the same as its geometry before the transition. As a result of this nonperturbative symmetry we expect that ρ (and consequently F) should be an even function of μ . Surprisingly, this expectation is apparently violated by the leading scaling behavior of ρ and F that we just determined in the WKB approximation. This apparent paradox will be resolved when we obtain the exact nonperturbative formula for ρ , which will indeed be an even function. The odd piece in the perturbative expansion is a consequence of the expansion in powers of $1/\mu$, which splits the exact formula into a perturbative and a nonperturbative piece, neither of which is even under (5.16).

5.2 The double-scaling limit

Having identified the correct scaling variable, it is now clear how to define the double-scaling limit of the M-theory vacuum. It is given by distributing N fermions such that they fill all the lowest energy levels up to a Fermi energy ε_F , and then taking the limit

$$N \rightarrow \infty, \quad \varepsilon_F \rightarrow 0, \quad \mu \equiv -N\varepsilon_F \text{ fixed.} \quad (5.17)$$

Thus, the rules for taking the double scaling limit of fermions in the M-theory vacuum turn out to be exactly the same as in noncritical string theory in 1 + 1 dimensions. The double-scaling limits leading to Type 0A, 0B or M-theory solutions differ only in the selection of how the N fermions occupy available energy levels, but not in how the $N \rightarrow \infty$ limit is taken.

5.3 The worldsheet cosmological constant and the string susceptibility

If this were a string theory, we would be interested in expressing the amplitudes in terms of the worldsheet cosmological constant Δ . In the matrix model, this cosmological constant can be defined via

$$\frac{\partial \Delta}{\partial \mu} = \pi \rho(\mu). \quad (5.18)$$

In string theory, a particularly important critical exponent is γ_{str} , known as the “string susceptibility” exponent [3]–[7]. It is usually defined via the leading scaling behavior of the vacuum energy of the matrix model in the double scaling limit,

$$F \sim \Delta^{2-\gamma_{\text{str}}} + \dots \quad (5.19)$$

In matrix models of noncritical strings, we are limited to backgrounds with spacetime dimension $d \leq 2$. These backgrounds have $\gamma_{\text{str}} \leq 0$, with the bound $\gamma_{\text{str}} = 0$ saturated for two-dimensional strings, or central charge c (or \hat{c}) equal to one. This is the famous “ $c = 1$ barrier” of the matrix model formulation of noncritical string theory.⁷

In noncritical M-theory, we can define Δ as in (5.18). In the leading WKB approximation, we get

$$\Delta \approx -\frac{\pi}{2\omega_0^2} \mu^2 + \dots \quad (5.20)$$

⁷The $c = 1$ barrier can perhaps be breached by considering supersymmetric noncritical strings with an exotic type of supersymmetry, [27].

Furthermore, we can introduce the ‘‘M-theory susceptibility’’ exponent γ_M , defined exactly as in string theory via eq. (5.19). As we have seen, in the M-theory vacuum $F \sim \mu^3 + \dots$, implying

$$\gamma_M = 1/2. \quad (5.21)$$

Thus, the noncritical M-theory vacuum is in the regime of values of the susceptibility exponent that is unattainable by matrix models of noncritical strings. This is yet another check that our M-theory is naturally interpreted as living beyond the $c = 1$ barrier, at the cost of not being a string theory anymore.

6. Exact vacuum energy in noncritical M-theory

Having defined our M-theory vacuum solution $|M, \mu\rangle$, we can now use the exact free-fermion description of the system to extract a wealth of physical information about $|M, \mu\rangle$ and its excitations. As an example, we will evaluate the exact vacuum energy of this solution.

As in the matrix models of noncritical string theory, the vacuum energy $F(\mu)$ is determined in terms of the exact density of states $\rho_M(\mu)$ via

$$\frac{\partial F}{\partial \mu} = \mu \rho_M(\mu). \quad (6.1)$$

The μ derivative of ρ_M can be expressed in the following, cutoff-independent integral representation,

$$\begin{aligned} \frac{\partial \rho_M}{\partial \mu} &= \sum_{q \in \mathbf{Z}} \frac{\partial \rho_{0A}}{\partial \mu} = \frac{1}{2\pi\omega_0\mu} \operatorname{Im} \int_0^\infty d\sigma e^{-i\sigma} \frac{\omega_0\sigma/\mu}{\sinh\{\omega_0\sigma/\mu\}} \sum_{q \in \mathbf{Z}} e^{-|q|\omega_0\sigma/\mu} \\ &= \frac{1}{2\pi\omega_0\mu} \operatorname{Im} \int_0^\infty d\sigma e^{-i\sigma} \frac{\omega_0\sigma/(2\mu)}{\sinh^2\{\omega_0\sigma/(2\mu)\}}. \end{aligned} \quad (6.2)$$

Recall that the scale ω_0 is related in Type 0A and 0B string theory to the string scale via $\omega_0 = 1/\sqrt{2\alpha'}$. We obtained the integral representation (6.2) by summing the Type 0A contributions [1, 18] from the sectors of all integer values of RR flux q .

Alternatively, this integral representation could be obtained in a manner closer to Type 0B, using the Cartesian coordinate representation of M-theory and the definition of the density of states via the resolvent of the one-particle Hamiltonian $h(p_i, \lambda_i)$,

$$\rho_M(\mu) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \operatorname{Im} \operatorname{Tr} \left(\frac{1}{h(p_i, \lambda_i) + \mu - i\epsilon} \right). \quad (6.3)$$

The resolvent can be easily evaluated, leading to

$$\begin{aligned} \langle \tilde{\lambda}_i | \frac{1}{h + \mu - i\epsilon} | \lambda_j \rangle &= i \int_0^\infty d\tau e^{-i\mu\tau} \left(\frac{i\omega_0}{2\pi \sinh(\omega_0\tau)} \right) \times \\ &\times \exp \left\{ \frac{i\omega_0[(\lambda^2 + \tilde{\lambda}^2) \cosh(\omega_0\tau) - 2\lambda\tilde{\lambda}]}{2 \sinh(\omega_0\tau)} \right\}. \end{aligned} \quad (6.4)$$

Upon evaluating the Gaussian integrals over λ_i and using (6.3), we obtain

$$\rho_M(\mu) = \frac{1}{4\pi} \operatorname{Re} \int_0^\infty d\tau e^{-i\mu\tau} \frac{1}{\sinh^2(\omega_0\tau/2)}. \quad (6.5)$$

Strictly speaking, this formula depends on a cutoff (i.e., is formally divergent near $\tau = 0$ and needs to be regulated), but in a way which is μ independent. Taking the derivative of (6.5) and rescaling the integration variable to $\sigma = \mu\tau$, we reproduce (6.2).

6.1 The weak coupling expansion

We see that the M-theory vacuum has a dimensionless parameter, μ/ω_0 . In string theory, this parameter would play the role of the inverse string coupling constant. Thus, in analogy with string theory, we first study the perturbation expansion in the powers of $1/\mu$.

6.1.1 Leading order

The leading term in the expansion of the density of states is μ -independent, and equal to

$$\begin{aligned} \frac{\partial\rho_M}{\partial\mu} &\approx -\frac{1}{\pi\omega_0^2} \int_0^\infty \frac{d\sigma}{\sigma} \sin\sigma e^{-\epsilon\sigma} + \mathcal{O}(1/\mu^2) \\ &= -\frac{1}{2\omega_0^2} + \mathcal{O}(1/\mu^2). \end{aligned} \quad (6.6)$$

This is to be contrasted with the $1/\mu$ expansion in two-dimensional string theory, where the leading term in $\partial\rho/\partial\mu$ goes as $1/\mu$, leading to the characteristic logarithmic behavior of $\rho(\mu)$.

6.1.2 Higher loops

In the $1/\mu$ expansion of the derivative of the density of states (6.2), only even powers of $1/\mu$ appear, and the term of order $2m$ (with $m = 1, 2, \dots$) is proportional to

$$\int_0^\infty d\sigma \sigma^{2m-1} \sin\sigma e^{-\epsilon\sigma}. \quad (6.7)$$

All such integrals vanish identically, implying that our perturbation series for $\partial\rho_M/\partial\mu$ terminates after the lowest, constant term, and we obtain

$$\frac{\partial\rho_M}{\partial\mu} \approx -\frac{1}{2\omega_0^2} + \text{possible nonperturbative terms.} \quad (6.8)$$

This in turn leads to the asymptotic expansion of the density of states

$$\rho_M(\mu) \approx -\frac{1}{2\omega_0^2} \mu + \frac{C}{\omega_0}, \quad (6.9)$$

where C is a nonuniversal dimensionless integration constant, to be discussed in section 6.3. Finally, this yields the perturbative formula for the vacuum energy, exact to all orders in powers of $1/\mu$,

$$F = \int^{-\mu} \nu \rho_M(\nu) d\nu \approx \frac{1}{2\omega_0^2} \int^{-\mu} \nu^2 d\nu = -\frac{1}{6\omega_0^2} \mu^3 + \frac{C}{2\omega_0} \mu^2 + \omega_0 C_0. \quad (6.10)$$

The integration constant C_0 represents a one-loop term. Unlike in noncritical string theory, where the one-loop term is proportional to $\log \mu$, C_0 in M-theory is μ independent, and can therefore be eliminated by a shift in the overall zero of the vacuum energy F . We will set $C_0 = 0$ from now on.

We found a dramatic simplification in the $1/\mu$ expansion of the vacuum energy of the M-theory vacuum, compared to its string theory counterparts (where all orders in $1/\mu$ are generically nonzero). The fact that the perturbative expansion terminates at one loop is a first hint that the theory may be topological, or at least exhibit a localization of the path integral similar to that of a topological theory. This will be further confirmed when we study the structure of nonperturbative corrections to the vacuum energy below.

6.1.3 Summation of higher-genus Type 0A contributions

From the point of view of string theory, this result can be reproduced by summing the asymptotic expansions of the Type 0A amplitudes order by order in $1/\mu$. The vacuum energy in Type 0A theory with RR flux q can also be expanded in the powers of the string coupling $1/\mu$, and the coefficients of this series are generically nonzero to all orders. It is instructive to see how they sum up to zero, order by order in $1/\mu$, when the summation over q is performed. In the derivative of the density of states, these terms of order $1/\mu^3$ and higher are (with the implicit ζ -function regularization)

$$\begin{aligned} \frac{\partial \rho}{\partial \nu} &= \frac{2}{\pi \omega_0^2} \sum_{k=1}^{\infty} \frac{k^2 \nu / \omega_0}{(k^2 + \nu^2 / \omega_0^2)^2} \\ &\approx \frac{2\omega_0}{\pi \nu^3} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} (m+1) (-1)^m k^{2m} \left(\frac{\omega_0}{\nu}\right)^{2m} \\ &\approx -\frac{2}{\pi \omega_0^2} \sum_{m=1}^{\infty} (-1)^m m \zeta(-2m) \left(\frac{\omega_0}{\nu}\right)^{2m+1}. \end{aligned} \tag{6.11}$$

Hence, the nonzero contributions from sectors of fixed q sum up, at order $(1/\mu)^{2m+1}$, to give $\zeta(-2m)$. Since $\zeta(-2m) = 0$ for $m = 1, \dots$, all terms $m = 1, \dots$ in this asymptotic expansion are identically zero.

6.1.4 Summation of the leading logs

It is similarly instructive to see how the leading μ^3 behavior of the vacuum energy in M-theory comes about from the summation of the leading $\mu^2 \log \mu$ terms in Type 0A at fixed q . (For simplicity, we set $\omega_0 = 1$ in this paragraph.)

Recall that at fixed $q \in \mathbf{Z}$, the density of states in Type 0A theory has an asymptotic expansion [1, 18, 19]

$$\rho_{0A}(\nu, q) \approx -\frac{1}{2\pi} \operatorname{Re} \log(|q| - i\nu) + \mathcal{O}(1/(|q| - i\nu)). \tag{6.12}$$

In M-theory, we fill all sectors with different values of q up to the common Fermi level. The leading term in the expansion of the density of states is then

$$\rho_M(\mu) = \sum_{q \in \mathbf{Z}} \rho_{0A}(\mu, q) \approx -\frac{1}{4\pi} \sum_{q \in \mathbf{Z}} \log(\mu^2 + q^2) + \dots \tag{6.13}$$

We are interested in summing these leading logs. We have

$$\begin{aligned}
 -\frac{1}{4\pi} \sum_{q \in \mathbf{Z}} \log(\mu^2 + q^2) &= -\frac{1}{4\pi} \log(\mu^2) - \frac{1}{2\pi} \sum_{q=1}^{\infty} \log(q^2(1 + \mu^2/q^2)) \\
 &= -\frac{1}{4\pi} \log(\mu^2) - \frac{1}{2\pi} \sum_{q=1}^{\infty} \log(q^2) - \frac{1}{2\pi} \log \prod_{q=1}^{\infty} (1 + \mu^2/q^2) \\
 &= -\frac{1}{4\pi} \log(\mu^2) - \frac{1}{2\pi} \log \left[\frac{\sinh(\pi\mu)}{\pi\mu} \right] + \dots \\
 &= -\frac{1}{2\pi} \log(\sinh(\pi\mu)) + \dots = -\frac{\mu}{2} - \frac{1}{2\pi} \log(1 - e^{-2\pi\mu}) + \dots \\
 &= -\frac{\mu}{2} + \dots \tag{6.14}
 \end{aligned}$$

where the “...” in (6.14) refer to divergent but μ -independent terms, and where in the final formula we also dropped all the terms nonperturbative in $1/\mu$.

Thus we see that the leading $\log \mu$ piece from the $q = 0$ sector is exactly offset by a contribution from $\log \sinh(\pi\mu)/\mu$ which originates in the sum over sectors with $q \neq 0$. Instead, the leading log is replaced by a term linear in μ , which also emerges from the sum over all q . Consequently, we end up with the M-theory scaling,

$$\rho_M(\mu) \sim \mu \equiv \kappa^{-2/3}, \tag{6.15}$$

predicted by the WKB argument of the previous subsection.

6.2 The strong coupling expansion

We now turn to the analysis of the nonperturbative corrections.

The integral representation for the derivative of the density of states can be expanded in the powers of μ :

$$\begin{aligned}
 \frac{\partial \rho_M}{\partial \mu} &\approx -\frac{1}{\pi\omega_0^2} \sum_{n=0}^{\infty} (-1)^n \frac{(2\mu/\omega_0)^{2n+1}}{(2n+1)!} \int_0^{\infty} d\tau \frac{\tau^{2n+2}}{\sinh \tau} \\
 &\approx -\frac{1}{\omega_0^2} \sum_{n=0}^{\infty} \frac{(2\pi\mu/\omega_0)^{2n+1} B_{2n+2}}{(2n+1)!}. \tag{6.16}
 \end{aligned}$$

Alternatively, this same result can be obtained by summing Type 0A contributions over all values of q . Indeed,

$$\begin{aligned}
 \frac{\partial \rho_M}{\partial \nu} &= \frac{2}{\pi\omega_0^2} \sum_{k=1}^{\infty} \frac{\nu/\omega_0^2}{k^2(1 + \nu^2/(k^2\omega_0^2))^2} \\
 &\approx \frac{2}{\pi\omega_0^2} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} (m+1)(-1)^m \frac{(\nu/\omega_0)^{2m+1}}{k^{2m+2}} \\
 &\approx -\frac{2}{\pi\omega_0^2} \sum_{m=1}^{\infty} m(-1)^m \zeta(2m) \left(\frac{\nu}{\omega_0}\right)^{2m-1}. \tag{6.17}
 \end{aligned}$$

The zeta function at positive even integers can be expressed in terms of the Bernoulli numbers,

$$\zeta(2n) = \frac{2^{2n-1} \pi^{2n} |B_{2n}|}{(2n)!}. \tag{6.18}$$

Using the fact that $B_{2n} = (-1)^{n+1} |B_{2n}|$ for $n = 1, \dots$, we get

$$\frac{\partial \rho_M}{\partial \nu} \approx -\frac{1}{\omega_0^2} \sum_{m=1}^{\infty} 2m \left(\frac{2\pi\nu}{\omega_0} \right)^{2m-1} \frac{B_{2m}}{(2m)!}, \tag{6.19}$$

reproducing (6.16).

6.3 Dependence on the cutoff and volume

So far, we have only considered the universal part of the density of states and the vacuum energy. Now we take a closer look at the possible cutoff dependence. Recall that in non-critical string theory, the vacuum energy is dependent on the cutoff Λ via the $\mu^2 \log(\Lambda/\mu)$ and $\log(\Lambda/\mu)$ terms in the string coupling expansion. These two terms have a clear physical interpretation: $\log(\Lambda/\mu)$ is the effective volume of the Liouville dimension, and the tree-level and one-loop terms in the vacuum energy are proportional to this volume.

The density of states in the M-theory vacuum is similarly cutoff-dependent. The proper way of defining the double-scaling limit of ρ involves first introducing a small- τ cutoff in the integral representation

$$\rho_M(\mu) = \frac{1}{4\pi} \operatorname{Re} \int_{1/\Lambda}^{\infty} d\tau e^{-i\mu\tau} \frac{1}{\sinh^2(\omega_0\tau/2)}, \tag{6.20}$$

and then taking Λ (which is proportional to \sqrt{N}) to infinity. In the previous subsections, we took advantage of the fact that the entire cutoff dependence of $\rho(\mu)$ is associated with the constant, μ -independent term in $\rho(\mu)$, and we simply evaluated the finite, universal quantity $\partial\rho/\partial\mu$. The leading, cutoff dependent term in ρ is given by

$$\frac{1}{4\pi\omega_0} \int_{1/\Lambda}^{\infty} d\tau \frac{1}{\sinh^2(\omega_0\tau/2)} = \frac{1}{2\pi\omega_0} \left(\coth\left(\frac{\omega_0}{2\Lambda}\right) - 1 \right) \approx \frac{\Lambda}{\pi\omega_0^2} + \dots, \tag{6.21}$$

where in the end we dropped all subleading terms in $1/\Lambda$.⁸ This constant term modifies the leading behavior of the exact density of states in the $1/\mu$ expansion to

$$\rho_M(\mu) \approx \frac{1}{2\omega_0^2} (-\mu + \Lambda) + \text{nonperturbative terms}. \tag{6.22}$$

Upon further integration, the cutoff-dependent term in ρ will give a Λ -dependent correction to our previous expression for the vacuum energy,

$$F \approx -\frac{\mu^3}{6\omega_0^2} + \frac{\Lambda\mu^2}{4\omega_0^2}. \tag{6.23}$$

⁸Throughout this paper, Λ represents a large, nonuniversal cutoff. Consequently, we will only keep track of the leading dependence on Λ , and systematically drop all the subleading nonuniversal terms in all the cutoff-dependent quantities.

In retrospect, we should have expected this Λ -dependent contribution to the leading behavior of the exact density of states, given the results of our WKB calculation in section 5.1, where the same μ -independent, Λ -dependent additive correction to $\rho \sim \mu$ was also found.

In analogy with noncritical string theory, it is natural to interpret Λ as a measure of the total volume of the system. We see that in the noncritical M-theory vacuum — just as in string theory — the volume dependence creeps in via the μ^2 term in the vacuum energy. Unlike in string theory, however, this cutoff dependence does not affect the leading, tree-level term, which in M-theory scales as μ^3 . The volume-dependent terms in the vacuum energy are now only subleading, of order $\kappa^{2/3}$ compared to the leading tree-level contribution. It is intriguing to recall that in critical heterotic M-theory [26], it is at this order where the twisted sector (described by Yang-Mills degrees of freedom at the boundary) starts contributing.

6.4 The exact formula

Thus, the strong coupling expansion of ρ_M in powers of μ results in a nontrivial series (6.19). This series can be summed as follows. Recall first that the Bernoulli numbers are usually defined via their generating function,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \tag{6.24}$$

Together with the elementary facts that $B_{2k+1} = 0$ for all $k = 1, 2, \dots$ while $B_0 = 1$ and $B_1 = -1/2$, this allows us to rewrite (6.19) as

$$\frac{\partial \rho_M}{\partial \nu} = -\frac{1}{\omega_0^2} \frac{\partial}{\partial \nu} \left(\frac{\nu}{e^{2\pi\nu/\omega_0} - 1} - \frac{B_1}{2\pi} 2\pi\nu \right) = -\frac{1}{\omega_0^2} \frac{\partial}{\partial \nu} \left(\frac{\nu}{2} + \frac{\nu}{e^{2\pi\nu/\omega_0} - 1} \right). \tag{6.25}$$

To further verify this, we now evaluate (6.2) directly, using a contour integral method while keeping track of the expected asymptotics in μ .

6.4.1 Evaluation by a contour integral

We are interested in

$$I \equiv \frac{1}{4\pi} \int_0^\infty dx \frac{e^{-i\mu x}}{\sinh^2(x/2)} \equiv \int_0^\infty dx I(x). \tag{6.26}$$

This integral can be evaluated as follows. The integrand $I(z)$, as a function in the complex plane, has an infinite series of double poles at $z = 2k\pi i$ for all $k \in \mathbf{Z}$. Furthermore, $I(z)$ is a quasi-periodic function along the imaginary axis,

$$I(x + 2\pi i) = e^{2\pi\mu} I(x). \tag{6.27}$$

Taking advantage of this quasi-periodicity of $I(z)$, we can close the contour as in figure 1, and obtain in the limit $L \rightarrow \infty$

$$2(1 - e^{2\pi\mu})I + \int_{C_0} I(z) dz + \int_{C_{2\pi i}} I(z) dz = \oint_C I(z) = 2\pi i \text{Res}_{2\pi i} I(z) = 2\mu e^{2\pi\mu}. \tag{6.28}$$

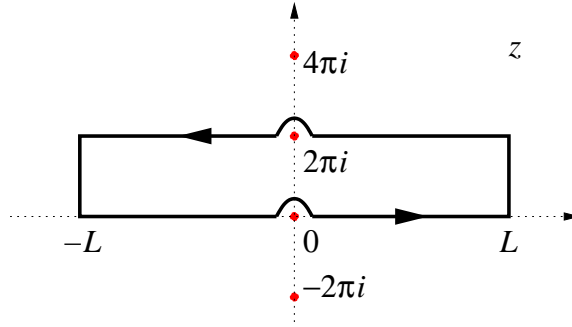


Figure 1: The integration contour C used to evaluate the exact density of states. The singularities are at $2k\pi i$, the contour encloses one of them — at $2\pi i$ — and the integral is evaluated in the limit of $L \rightarrow \infty$.

where C_0 and $C_{2\pi i}$ are the two semicircles of radius $\epsilon \sim 1/\Lambda$ around the poles at 0 and $2\pi i$. The contributions from C_0 and $C_{2\pi i}$ are divergent, and equal to

$$\begin{aligned} \int_{C_0} I(z) &= -\frac{2}{\pi\epsilon} - \pi i \operatorname{Res}_0 I(z) + \mathcal{O}(\epsilon), \\ \int_{C_{2\pi i}} I(z) &= \frac{2}{\pi\epsilon} e^{2\pi\mu} + \pi i \operatorname{Res}_{2\pi i} I(z) + \mathcal{O}(\epsilon). \end{aligned} \quad (6.29)$$

This yields

$$2(1 - e^{2\pi\mu})I = \pi i \operatorname{Res}_0 I(z) + \pi i \operatorname{Res}_{2\pi i} I(z) + (1 - e^{2\pi\mu})\frac{2}{\pi\epsilon}, \quad (6.30)$$

and finally (after identifying $\epsilon \sim 1/\Lambda$)

$$I = -\frac{\mu}{2 \tanh(\pi\mu)} + \frac{\Lambda}{2}. \quad (6.31)$$

Restoring ω_0 in (6.31) we get

$$\rho_M(\mu) = -\frac{\mu/\omega_0^2}{2 \tanh(\pi\mu/\omega_0)} + \frac{\Lambda}{2\omega_0^2}. \quad (6.32)$$

This formula is exact, and matches the results of our summation (6.25). One can easily check that it has the correct asymptotics to match both the weak coupling and the strong coupling expansion. Notice that the exact density of states (6.32) is now even under $\mu \rightarrow -\mu$, despite the fact that the leading behavior in the asymptotic expansion in $1/\mu$ is odd, $\sim \mu^3$. This is the promised resolution of the puzzle mentioned in section 5.1. We see that the leading perturbative term being odd in μ is an artifact of splitting the exact formula into the perturbative and the nonperturbative part, neither of which are separately even under $\mu \rightarrow -\mu$.

The exact vacuum energy is then

$$\begin{aligned} F = & -\frac{1}{2\omega_0^2} \int^{-\mu} d\nu \left\{ \frac{\nu^2}{\tanh(\pi\nu)} - \Lambda\nu \right\} = -\frac{1}{6\omega_0^2} \mu^3 + \frac{\Lambda}{4\omega_0^2} \mu^2 \\ & - \frac{1}{2\pi\omega_0} \mu^2 \log(1 - e^{-2\pi\mu/\omega_0}) + \frac{1}{2\pi^2} \mu \operatorname{Li}_2(e^{-2\pi\mu/\omega_0}) + \frac{\omega_0}{4\pi^3} \operatorname{Li}_3(e^{-2\pi\mu/\omega_0}) \end{aligned} \quad (6.33)$$

In the last expression, the first two terms correspond to the perturbative contribution in the $1/\mu$ expansion, while the terms involving the log and the polylogarithms⁹ all represent nonperturbative corrections.

In section 5.3 we determined that the M-theory analog of the string susceptibility in the M-theory vacuum is equal to $1/2$. This critical exponent measures the leading scaling of the vacuum energy with Δ , a scaling variable that in string theory can be identified with the worldsheet cosmological constant. Indeed, Δ is the continuum limit of the bare cosmological constant, which counts the number of vertices in the triangulation of the random surface in the matrix model. Δ in the M-theory vacuum can also be exactly evaluated, leading to

$$\Delta(\mu) = -\frac{\pi}{2} \left[\frac{1}{\omega_0^2} \mu^2 + \frac{2}{\omega_0} \mu \log(1 - e^{-2\pi\mu/\omega_0}) - \text{Li}_2(e^{-2\pi\mu/\omega_0}) \right] + \frac{\pi\Lambda}{2\omega_0^2} \mu. \quad (6.34)$$

We have again dropped a possible nonuniversal Λ -dependent constant term independent of μ .

6.5 The exact formula in the weak coupling expansion

The compact formula (6.33) for the exact vacuum energy can be rewritten in a more illuminating way by re-expanding in $1/\mu$, into an infinite sum of instanton-like terms,

$$F = \omega_0 \left\{ -\frac{1}{6\omega_0^3} \mu^3 + \frac{\Lambda}{4\omega_0^3} \mu^2 + \sum_{k=1}^{\infty} \left(\frac{1}{2\pi\omega_0^2 k} \mu^2 + \frac{1}{2\pi^2\omega_0 k^2} \mu + \frac{1}{4\pi^3 k^3} \right) e^{-2\pi k\mu/\omega_0} \right\}. \quad (6.35)$$

This formula exhibits several noteworthy features:

- (1) We again find that the weak-coupling expansion is in fractional powers of the natural loop counting parameter κ^2 , the basic unit of the expansion being $\kappa^{2/3}$.
- (2) The strength of all nonperturbative effects (i.e., the “instanton action”) is controlled by $1/\mu \equiv \kappa^{2/3}$, leading to their scaling as $\sim e^{-A\kappa^{-2/3}}$ for some constant A .
- (3) In the vicinity of each instanton, the perturbative expansion involves terms of order $\kappa^{2/3}$, $\kappa^{4/3}$, and κ^0 (which corresponds to one loop). As in the case of the perturbative part of the vacuum energy, all higher orders terms vanish. This is again strongly indicative of localization phenomena and an underlying topological symmetry of the theory. It is intriguing, however, that this topological nature of the theory is compatible with the anticipated presence of a propagating degree of freedom: The M-theory analog of the massless stringy tachyon.¹⁰ In this respect, the vacuum energy of the M-theory vacuum exhibits features reminiscent of a holographic field theory [28, 29].

⁹Our notation for the polylogarithms is such that $\text{Li}_\nu(z) = \sum_{k=1}^{\infty} z^k/k^\nu$ for $|z| < 1$.

¹⁰A simple heuristic argument for the presence of a propagating degree of freedom can be given. In the fermion language, the M-theory vacuum has a smooth semiclassical Fermi surface, of codimension one in phase space. This Fermi surface can fluctuate and its small fluctuations correspond to gapless bosonic excitations, implying the presence of a propagating degree of freedom.

- (4) Unlike for the one-loop-exact perturbative contribution, the instanton measure contributions start only at the subleading order $\kappa^{2/3}$ compared to the tree-level term. The analogy with critical heterotic M-theory [26] suggests that the instantons could be related to twisted sectors of the theory, with their characteristic subleading behavior $\sim \kappa^{2/3}$.
- (5) Instead of calculating the vacuum energy F , one may be interested in the free energy $\Gamma(\mu)$, defined via the Legendre transform of F as a function of Δ :

$$\Gamma(\mu) = \mu\Delta - F(\Delta). \tag{6.36}$$

The exact formula for $\Gamma(\mu)$ can be easily obtained from (6.34) and (6.35).

7. Analogy with the Debye model of phonons in solids

The universal part of our exact formula (6.33) for the vacuum energy in the M-theory vacuum can be rewritten as

$$F = -\frac{1}{2\omega_0^2} \int^{-\mu} \nu^2 \left(\frac{1}{\exp(2\pi\nu/\omega_0) - 1} + \frac{1}{2} \right) d\nu. \tag{7.1}$$

This expression is strongly suggestive of underlying bosonic degrees of freedom. At first, it appears that (7.1) represents the energy of a thermal bosonic system, with the density of states yielding Planck's black body radiation formula at an effective temperature of order one in string units,

$$T_{\text{eff}} = \frac{\omega_0}{2\pi}. \tag{7.2}$$

However, (7.1) also exhibits an effective cutoff on the available frequencies, which is absent in the Planck black body formula. Upon closer inspection, (7.1) turns out to be more closely analogous to another famous bosonic system: the Debye model of phonon excitations in a solid at temperature (7.2).

7.1 The Debye model

The Debye model of the thermodynamic properties of solids [30, 31] was originally designed to explain the behavior of the specific heat of solids at low temperatures. It was proposed as an improvement of the somewhat less successful Einstein model, in which the atoms in the solid were simply treated as independent harmonic oscillators. In Debye's model, the crystal consists of a fixed number $\sim \mathcal{N}$ of atoms, assumed to behave as a system of coupled harmonic oscillators, with a fixed number \mathcal{N} of normal modes of the phonon spectrum. The total energy of the system at temperature T is given by an integral over all frequencies,

$$E = \int_0^{\omega_D} \omega \rho_D(\omega) \left(\frac{1}{e^{\omega/T} - 1} + \frac{1}{2} \right) d\omega, \tag{7.3}$$

with $\rho_D(\omega)$ the density of states of the system. The finiteness of the total number of atoms imposes a limit ω_D on the maximum attainable frequency of the normal modes. This

limiting frequency, referred to as the Debye frequency, is set by the total number \mathcal{N} of normal modes in the crystal,

$$\int_0^{\omega_D} \rho_D(\omega) d\omega = \mathcal{N}. \tag{7.4}$$

As a model of realistic crystals, the Debye model is based on several rather drastic simplifying assumptions about the system. Firstly, the phonon dispersion relation is assumed isotropic and strictly relativistic. Furthermore, the assumed absence of anharmonic terms in the phonon system is equivalent to ignoring the possibility of crystal melting at high enough temperature. Lastly, the density of states $\rho_D(\omega)$ is assumed to be a smooth function $\rho_D(\omega) \sim \omega^{d-1}$, with d the number of spatial dimensions, up to the sharp cutoff at $\omega = \omega_D$. For example, in two spatial dimensions, one would have

$$\rho_D(\omega) = \frac{V}{2\pi} \omega, \tag{7.5}$$

where V is the volume of the system, and the speed of sound has been set equal to one.

All of these assumptions would have to be modified in a realistic crystal. In contrast, as we are now going to see, the exact calculations in M-theory are compatible with all of the above assumptions, and in this sense, the Debye analogy for noncritical M-theory is exact.

7.2 The analogy

It is easy to see that our formula (7.1) for the exact vacuum energy of M-theory is precisely of the Debye form (7.3), with the following dictionary between the two descriptions of the system:

- (1) The chemical potential in the bosonic system is equal to zero. This means that the bosonic quanta can be created and annihilated, and that the total number of bosons is not fixed.
- (2) The perturbative piece in $\rho(\mu)$ corresponds to the zero-point energy of the Debye crystal. The nonperturbative terms in $\rho(\mu)$ sum up to the Planck thermal factor.
- (3) The double-scaled Fermi energy μ plays the role of the Debye frequency ω_D .
- (4) The effective temperature of the crystal is given by (7.2). It is set in string units and cannot be varied, at least in this vacuum of noncritical M-theory.
- (5) The Debye density of states is proportional to ω . Thus, the system is effectively 2 + 1 dimensional, as suggested by its M-theory interpretation as the M-theory vacuum.
- (6) The formula is consistent with the exact relativistic dispersion relation of the phonons, and with the relativistic density of states $\rho_D(\omega) = V\omega/2\pi$.
- (7) The total volume V of the Debye crystal is proportional to $1/\omega_0^2$. However, the surprise lies in the overall sign of this volume, which comes out negative!

This last point can be better understood as follows. Note first that the effective Debye density of states $\rho_D(\omega)$ that appears in (7.3) can be identified with the leading perturbative term in our density of states $\rho(\mu)$:

$$\rho(\omega) \sim \rho_D(\omega) + \text{nonperturbative terms.} \tag{7.6}$$

Recalling now the relation between the scaling variable Δ and the density of states,

$$\Delta = \pi \int^{-\mu} \rho(\nu) d\nu, \tag{7.7}$$

the leading perturbative term in Δ is found to be related, via (7.4), to the total number of atoms in the Debye solid:

$$\Delta \sim \mathcal{N} + \text{nonperturbative terms.} \tag{7.8}$$

In string theory, the scaling variable Δ was interpreted as the worldsheet cosmological constant, since its discretized matrix-model version counts the number of plaquettes in the random triangulation of the worldsheet. Surprisingly, we see that even in noncritical M-theory, Δ can be interpreted as an object that counts the number of constituents, now of the Debye crystal.

7.3 Reintroducing the cutoff: the melting crystal interpretation

The Debye analogy is almost precise, except that — as we have just seen — it seems to lead to the rather embarrassing prediction of a negative volume for the Debye crystal. This problem can be remedied by reintroducing the dependence on the cutoff Λ in the system.

We have seen in our exact evaluation of Δ in section 6.4 that when we keep track of the cutoff dependence, Δ gets a large positive contribution proportional to $\Lambda\mu$,

$$\Delta(\mu) = -\frac{\pi}{2\omega_0^2}\mu^2 + \frac{\pi\Lambda}{2\omega_0^2}\mu. \tag{7.9}$$

As we have just argued, in the Debye model analogy, Δ counts the effective number of atoms in the Debye crystal. Hence, (7.9) shows that in the thermodynamic limit of large Λ , we effectively have a large Debye crystal whose number of atoms is measured by the cutoff Λ . The negative sign in front of the leading μ^2 term in Δ is now easily understood: A number of atoms, measured by μ , has been removed from the large crystal.¹¹ μ now represents the *lowest* frequency in the system, confirming that a small number of atoms has been *removed* from the Debye solid. Effectively, μ measures the size of a small hole in a big sample of the Debye crystal. This picture is superficially reminiscent of the recently found correspondence between topological strings and the statistical mechanics of a classical melting crystal [32].

¹¹In fact, this is closely analogous to the behavior of noncritical string theory, where the available volume of the Liouville dimension is measured by $\log(\Lambda/\mu)$. One can think of Λ as setting the size of the Liouville dimension in the weakly coupled asymptotic region. μ is then associated with the Liouville wall. At weak string coupling $\mu \gg 1$, Liouville wall effectively subtracts the available volume from the total volume set by Λ , similarly to the behavior we have observed in noncritical M-theory.

Having interpreted the cutoff Λ as the quantity that sets the total number of atoms in the Debye system, we can in fact sharpen the relation between Δ , \mathcal{N} and N even further. The cutoff-dependent terms in Δ will include a μ -independent constant, which we have been ignoring so far as nonuniversal. Restoring this term, we get

$$\Delta \sim \frac{\pi}{2\omega_0^2}(-\mu^2 + \Lambda^2) + \dots \tag{7.10}$$

The Λ^2 term can be thought of as coming from the lower integration bound in the definition of Δ in (7.7). Recalling now that in the large N limit, the nonuniversal cutoff Λ scales as \sqrt{N} , we obtain from (7.10) that

$$\mathcal{N} \sim N. \tag{7.11}$$

Thus, we find that the number of atoms in the Debye solid is effectively related to the number of fermions in the Fermi liquid.

7.4 Solution with two Fermi surfaces as a universal melting crystal

The reinterpretation of μ as a parameter measuring the number of atoms removed from a Debye crystal of size set by the cutoff Λ is pleasing, but the downside of this interpretation is in its reliance on the nonuniversal cutoff Λ . In particular, it would be desirable to have more detailed information about the bulk of the system. For example, we would like to know whether the large crystal is at the same temperature as the atoms removed from it.

The dependence of the Debye interpretation on the cutoff can be eliminated by considering a small modification of our construction of the M-theory vacuum state. Instead of using the nonuniversal cutoff Λ to provide the environment, introduce two Fermi levels, μ_{\pm} , with $\mu_+ < \mu_-$, and fill the Fermi sea only between μ_+ and μ_- . Hence, μ_+ and μ_- are the top and the bottom of the Fermi sea, respectively. This state is again interpreted in terms of the double-scaling limit: Define $\mu_{\pm} = -N\varepsilon_F^{\pm}$, and take the limit $N \rightarrow \infty$ and $\varepsilon_F^{\pm} \rightarrow 0$ while keeping μ_{\pm} fixed.

In this modified state $|M, \mu_+, \mu_- \rangle$, the universal part of the vacuum energy is

$$F = -\frac{2}{\omega_0^2} \int_{-\mu_-}^{-\mu_+} d\nu \nu^2 \left(\frac{1}{\exp(2\pi\nu/\omega_0) - 1} + \frac{1}{2} \right), \tag{7.12}$$

while Δ is

$$\Delta(\mu_+, \mu_-) = \frac{\pi}{2\omega_0^2}(\mu_-^2 - \mu_+^2) + \dots, \tag{7.13}$$

where the “...” stand for all the nonperturbative and nonuniversal terms. (7.12) is indeed the Debye result for the free energy of a crystal of size set by μ_- , with a portion of the crystal measured by μ_+ removed. If $\mu_- \gg \mu_+$, we have a small hole in a big Debye crystal. The dependence of all quantities on μ_{\pm} is universal. The system is at finite temperature of order one in string units, $T_{\text{eff}} = \omega_0/2\pi$.

It is natural to suspect that the bosonic features of the vacuum energy in the noncritical M-theory vacuum are related to the anticipated bosonization in terms of a collective degree of freedom, which should represent the M-theory lift of the massless tachyon of noncritical string theory. This connection, and the entire Debye analogy, deserves further study.

8. Observables and symmetries

Having discussed properties of a specific M-theory solution in the previous sections, we now address several more conceptual aspects of noncritical M-theory, which should find applications to a broader class of solutions. For the rest of the paper, we will set $\omega_0 = 1$.

8.1 Observables

We have seen that the exact vacuum energy has a very interesting structure, suggesting an underlying symmetry reminiscent of topological localization.

Despite appearances, and the suggestive simplicity of the exact vacuum energy, the M-theory vacuum still contains propagating degrees of freedom. The existence of a Fermi surface suggests that at least one field-theory degree of freedom is present. In string theory, the fluctuations of the Fermi surface correspond to the massless modes of the theory, i.e., the tachyon (and, in Type 0B, also the RR scalar). Motivated by how the tachyon emerges from the matrix models of two-dimensional string theories (see, e.g., [3]), we can define a set of natural observables given by the density of eigenvalues $\rho(t, \lambda_i) = \Psi^\dagger \Psi(t, \lambda_i)$ [33], or, more conveniently, by the inverse Laplace-like transform of ρ with respect to the eigenvalue coordinates,

$$\mathcal{O}_0(t, w_i) = \int d\lambda_1 d\lambda_2 e^{-w_1 \lambda_1 - w_2 \lambda_2} \Psi^\dagger \Psi(t, \lambda_i), \quad (8.1)$$

or the Fourier transform with respect to t ,

$$\mathcal{O}_0(\omega, w_i) = \int dt e^{i\omega t} \mathcal{O}_0(t, w_i). \quad (8.2)$$

Lessons learned in two-dimensional string theory lead us to anticipate that the field \mathcal{O}_0 should be the M-theory analog of the massless tachyon field. Indeed, it is this collective bosonic field that represents the fluctuations of the Fermi surface in circumstances where the latter is nicely defined. An even better representation of the observables is

$$\begin{aligned} \mathcal{O}(t, \ell, \phi) &= \int_{\sqrt{2\mu}}^{\infty} d\lambda e^{-\ell\lambda} \Psi^\dagger \Psi(t, \lambda, \phi) \\ &= \int_0^{\infty} d\lambda e^{-\sqrt{2\mu}\ell \cosh \tau} \Psi^\dagger \Psi(t, \tau, \phi), \end{aligned} \quad (8.3)$$

where we have introduced τ via $\lambda = \sqrt{2\mu} \cosh \tau$ in order to shift the lower integration bound to zero. These formulas are very reminiscent of the bosonization of nonrelativistic fermions in higher dimensions [34, 35], where the bosonization is in terms of a collection of 1 + 1 dimensional bosons parametrized by the angle ϕ on the Fermi surface, which plays the role of an internal index.

The natural correlation functions to calculate are the n -point functions

$$\langle M, \mu | \prod_{k=1}^n \mathcal{O}(t_k, \ell_k, \phi_k) | M, \mu \rangle. \quad (8.4)$$

They can again be evaluated exactly (in principle), using the techniques developed in the matrix models of noncritical strings [15, 37, 38]. We leave a detailed analysis for the future.

8.2 Symmetries

Our noncritical M-theory is an exactly solvable system, with an infinite dimensional symmetry algebra generalizing the famous w_∞ symmetries and ground ring structure of two-dimensional strings [39].

Recalling the classical equations of motion

$$\dot{p}_i = \lambda_i, \quad \dot{\lambda}_i = p_i, \quad (8.5)$$

the conserved charges can be built out of four building blocks (interpreting t again as a real time coordinate),

$$\begin{aligned} a_1 &= \frac{1}{\sqrt{2}}(p_1 + \lambda_1)e^{-t}, & b_1 &= \frac{1}{\sqrt{2}}(p_1 - \lambda_1)e^t, \\ a_2 &= \frac{1}{\sqrt{2}}(p_2 + \lambda_2)e^{-t}, & b_2 &= \frac{1}{\sqrt{2}}(p_2 - \lambda_2)e^t. \end{aligned} \quad (8.6)$$

The full symmetry algebra \mathcal{W} is generated by products of non-negative integral powers of a_i, b_i . Hence, a basis in \mathcal{W} is given by

$$W_{m_1 m_2 n_1 n_2} = a_1^{m_1} a_2^{m_2} b_1^{n_1} b_2^{n_2}, \quad m_i, n_i = 0, 1, \dots \quad (8.7)$$

The commutation relations are defined via the elementary Poisson brackets,

$$[a_i, b_j] = -\delta_{ij}, \quad [a_i, a_j] = [b_i, b_j] = 0. \quad (8.8)$$

Note that the Hamiltonian and the angular momentum are both bilinear combinations of a_i and b_i :

$$H = W_{1010} + W_{0101}, \quad J = W_{0110} - W_{1001}. \quad (8.9)$$

The elements in \mathcal{W} at most bilinear in a_i and b_i form a closed finite-dimensional subalgebra \mathcal{W}_0 of the full infinite symmetry algebra \mathcal{W} of the system.

In a typical solution, some symmetries from \mathcal{W} or \mathcal{W}_0 respect the Fermi surface, while others are broken by the solution. For example, our M-theory vacuum is preserved by just four (out of the total number of ten) quadratic charges: $W_{1010}, W_{1001}, W_{0110}$ and W_{0101} . They form the algebra of $SO(2, 1) \times U(1)$, with the Abelian generator corresponding to the Hamiltonian that defines the Fermi surface.

8.2.1 Massless modes vs. symmetries

In a given solution of noncritical M-theory, the massless bosonic modes are closely related to the existence of the Fermi surface. It is tempting to speculate that these massless modes should be interpreted as the Goldstone modes of the symmetries in \mathcal{W} that have been broken by the Fermi surface. Indeed, the bosonic fluctuations of the Fermi surface have been interpreted as Goldstone modes of broken symmetries in the condensed matter context (see, e.g., [34]).

If this view is correct, the states that exhibit higher degrees of symmetry should have fewer massless modes. The M-theory vacuum indeed exhibits a larger symmetry than the

Type 0A or 0B solutions. This larger degree of symmetry could explain the apparent topological features of the exact vacuum energy, compared to the less symmetric Type 0A or 0B string vacua. In this sense, the M-theory vacuum is closer than the string vacua to exposing the full symmetry of the theory.¹² This hypothetical Goldstone boson interpretation of the massless tachyon seems further supported by our interpretation of the vacuum energy in the M-theory vacuum in terms of the Debye phonons.

9. Semiclassical spacetime physics as hydrodynamics of the Fermi liquid

As we have argued, only the solutions of noncritical M-theory that can be bosonized in terms of hydrodynamic degrees of freedom are expected to admit a conventional semiclassical spacetime description. In this section, we develop a formalism — closely parallel to a similar framework in noncritical string theory [40, 41] — which allows us to search systematically for such hydrodynamic solutions of noncritical M-theory. Intuitively, the hydrodynamic states are those states that can be described by a semiclassical Fermi surface. In the semiclassical limit, the Fermi surface satisfies its own hydrodynamical equations of motion. Solving those equations directly is an efficient way of finding solutions of M-theory which admit a hydrodynamic description by design.

9.1 Classical equations of motion for the Fermi surface

The classical equations of motion for the Fermi surface in noncritical M-theory can be derived using the methods developed in noncritical string theory. In the classical limit, the Fermi surface is a (possibly time-dependent) hypersurface in the four-dimensional phase space of the system. The location of the Fermi surface in phase space can be described, for example, by choosing p_1 as the dependent variable,

$$p_1 \equiv P(x, y, w, t). \tag{9.1}$$

Here we have relabeled $\lambda_1 \equiv x$, $\lambda_2 \equiv y$, and $p_2 \equiv p_y \equiv w$, and have Wick-rotated t back to real time. Repeating the steps used in noncritical string theory, one can show that this function P satisfies the following classical equation of motion,

$$\partial_t P = x - P\partial_x P - w\partial_y P - y\partial_w P. \tag{9.2}$$

Sometimes it is convenient to use an alternative equation for the Fermi surface in the polar coordinates, in which the phase space is parametrized by r, ϕ and their canonically conjugate momenta p_r and $p_\phi \equiv J$. As our dependent variable to describe the Fermi surface, we can choose $p_r \equiv \mathcal{P}(r, \phi, p_\phi, t)$. The equations of motion for \mathcal{P} are

$$\partial_t \mathcal{P} = r + \frac{p_\phi^2}{r^3} - \mathcal{P}\partial_r \mathcal{P} - \frac{u}{r^2}\partial_\phi \mathcal{P}. \tag{9.3}$$

In the rest of this section, we will study several time-independent solutions of the theory, leaving time-dependent solutions for section 10. Needless to say, our selection of solutions is just a small sampling.

¹²Note, however, that any nontrivial Fermi surface will always break at least *some* of the \mathcal{W} symmetry, and it is thus not clear whether the theory has a ground state in which the entire underlying symmetry is unbroken.

9.2 Vacua with q as the scaling variable

The conventional Fermi surface that defined our M-theory vacuum in much of this paper,

$$\frac{p_r^2}{2} + \frac{p_\phi^2}{2r^2} - \frac{r^2}{2} = -\mu \tag{9.4}$$

satisfies the equation of motion, with

$$\mathcal{P}(r, \phi, u, t) = \sqrt{r^2 - \frac{p_\phi^2}{r^2} - 2\mu}. \tag{9.5}$$

So does a Fermi surface given by filling up to a fixed value of another conserved quantity, the angular momentum:

$$p_\phi = q. \tag{9.6}$$

However, the fluctuations around this surface are inconveniently parametrized in our representation of the Fermi surface by \mathcal{P} . We revert to the Cartesian coordinates, where this same surface is parametrized by

$$p_1\lambda_2 - p_2\lambda_1 = q, \tag{9.7}$$

leading to

$$P(x, y, w, t) = \frac{q + wx}{y}. \tag{9.8}$$

This satisfies the classical equation of motion for the Fermi surface in the Cartesian coordinate representation. We expect this solution to be related to two-dimensional string backgrounds with q as the scaling variable [16, 17, 42, 20] or to AdS_2 backgrounds [43, 44].

9.2.1 Duality to thermofield dynamics in the rightside-up harmonic potential

It turns out that a simple canonical transformation of the variables of our model maps our system to the thermofield dynamics of second-quantized fermions in the rightside-up harmonic potential.

The Fermi surface that fills all sectors up to a fixed q can be rewritten as follows. Define

$$\begin{aligned} x' &= \frac{1}{\sqrt{2}}(x + p_y), & p_{x'} &= \frac{1}{\sqrt{2}}(p_x - y), \\ y' &= \frac{1}{\sqrt{2}}(y + p_x), & p_{y'} &= \frac{1}{\sqrt{2}}(p_y - x). \end{aligned} \tag{9.9}$$

In these new variables, the Fermi surface is

$$\frac{1}{2} [p_{x'}^2 + (x')^2 - p_{y'}^2 - (y')^2] = -q. \tag{9.10}$$

This is simply a system consisting of two regular rightside-up harmonic oscillators, with a relative sign between the two Hamiltonians. Such a combination of two copies of the same Hamiltonian with a relative minus sign defines the real-time thermofield dynamics of the system (see, e.g., [45]–[49] for some background). Hence, we find the rather surprising

result, that our noncritical M-theory is dual to the thermofield dynamics of double-scaled fermions in the rightside-up harmonic oscillator potential.

A double-scaling limit of 1 + 1 dimensional fermions in the rightside-up harmonic potential has been studied, as the nonperturbative definition of a somewhat exotic version of $c = 1$ string theory [50, 51], (see also [52, 53] for another possible viewpoint). Here we see that this string theory is naturally embedded into our framework of noncritical M-theory. Indeed, the string theories of [50, 51] can be obtained as solutions by repeating the steps of section 4 in the primed variables. For example, filling only the states with a fixed value of ν'_2 will produce the string theory of the rightside-up harmonic oscillator studied in [50, 51].

Thermofield dynamics of a given system is not defined just by the doubling of the degrees of freedom and specifying the Hamiltonian. An important part of the definition is the preparation of an entangled vacuum state. In our case, this thermal state is

$$\sum e^{-E_\Phi/T} |\Phi\rangle \otimes |\tilde{\Phi}\rangle, \tag{9.11}$$

where the sum is performed over all quantum states $|\Phi\rangle$ of the second-quantized rightside-up harmonic oscillator, with E_Φ the energy of $|\Phi\rangle$). In accord with the philosophy of section 3.3, this thermal state of the thermofield dynamics of the rightside-up harmonic oscillator will be on the moduli space of all solutions of noncritical M-theory.

We also note in passing that if one performs the particle-hole duality on just one of the two upside-down oscillators that define noncritical M-theory, the Hamiltonian becomes that of the thermofield dynamics of one upside-down harmonic oscillator.

9.3 A family of stationary solutions

Clearly, a bigger class of time-independent classical solutions is obtained by combining the two conserved quantities, E and J , and postulating a Fermi surface

$$\frac{1}{2} (p_x^2 + p_y^2 - x^2 - y^2) + \Omega (p_x y - p_y x) = -\mu, \tag{9.12}$$

where Ω is a constant parameter. Since Ω serves as the chemical potential for the conserved angular momentum J , it can be interpreted as the angular velocity, leading to a simple interpretation of this solution as uniformly rotating. The vacuum energy of this state can again be evaluated exactly, as follows. In the polar coordinate representation of the model, the Fermi surface (9.12) can be viewed in each sector of fixed $J = q$ as the Type 0A theory with RR flux q and the Fermi sea filled up to a q -dependent Fermi level, effectively replacing μ by $\mu + \Omega q$. The summation over all values of q then leads to

$$\begin{aligned} \frac{\partial \rho(\mu, \Omega)}{\partial \mu} &= \frac{1}{2\pi\omega_0} \text{Im} \int_0^\infty d\tau \sum_{q \in \mathbf{Z}} e^{-i(\mu + \Omega q)\tau} \frac{\omega_0 \tau}{\sinh(\omega_0 \tau)} e^{-|q|\omega_0 \tau} \\ &= \frac{1}{2\pi\omega_0} \text{Im} \int_0^\infty d\tau e^{-i\mu\tau} \frac{\omega_0 \tau}{\cosh(\omega_0 \tau) - \cos(\Omega\tau)}, \end{aligned} \tag{9.13}$$

where we have temporarily restored the dependence on ω_0 . Integrating (9.13) once, we get

$$\rho(\mu, \Omega) = \frac{1}{2\pi} \text{Re} \int_0^\infty d\tau e^{-i\mu\tau} \frac{1}{\cosh(\omega_0 \tau) - \cos(\Omega\tau)}. \tag{9.14}$$

This again requires a cutoff at the lower integration limit $\tau \sim 0$.

In this family of solutions, the two conserved time-independent charges H and J have been essentially put on an equal footing. The main difference between them is that one of them is compact and the other one is not. In string theory, the μ and q are related to the string coupling and the RR flux, respectively, but from the higher-dimensional vantage point of noncritical M-theory they are much more closely related. We believe that this M-theory perspective may be at the core of some of the surprising patterns observed recently in the behavior of two-dimensional strings in [54].

9.4 A twisted M-theory state

There is a simple variation of the M-theory state, which illustrates several interesting points. This state $|\tilde{M}, \mu\rangle$ is defined by filling all states in sectors with even angular momentum q up to a Fermi surface $-\mu$, while filling all sectors with odd q down to $-\mu$:

$$\begin{aligned}
 a_q(\nu) |\tilde{M}, \mu\rangle = 0 & \quad \text{for } \begin{cases} \nu > -\mu, & q \text{ even,} \\ \nu < -\mu, & q \text{ odd,} \end{cases} \\
 a_q^\dagger(\nu) |\tilde{M}, \mu\rangle = 0 & \quad \text{for } \begin{cases} \nu < -\mu, & q \text{ even,} \\ \nu > -\mu, & q \text{ odd.} \end{cases}
 \end{aligned}
 \tag{9.15}$$

The calculation of the vacuum energy goes through as in the case of $|\text{M}, \mu\rangle$, with an additional $(-1)^q$ weighing the contribution of each sector of fixed q . This will change the density of states to

$$\rho_{\tilde{M}} = \frac{1}{4\pi} \text{Re} \int_0^\infty d\tau e^{-i\mu\tau} \frac{1}{\cosh^2(\omega_0\tau/2)}.
 \tag{9.16}$$

This integral can again be evaluated exactly,¹³ leading to

$$\rho_{\tilde{M}} = \frac{1}{2\pi\omega_0^2} \frac{\mu}{\sinh(\pi\mu/\omega_0)}.
 \tag{9.17}$$

This solution exhibits some interesting points:

- Unlike in the case of the M-theory state $|\text{M}, \mu\rangle$, the density of states and the vacuum energy of the twisted M-theory state are cutoff independent. Moreover, the leading term $\sim \mu$ in the $1/\mu$ expansion of the density of states is absent.
- In fact, the expression is fully nonperturbative in the $1/\mu$ expansion. The exact vacuum energy consists of an infinite series of nonperturbative terms,

$$F = -\frac{1}{\pi^4\omega_0^2} \sum_{k=0}^\infty \left(\frac{1}{2k+1} \left(\frac{\pi\mu}{\omega_0} \right)^2 + \frac{2}{(2k+1)^2} \frac{\pi\mu}{\omega_0} + \frac{2}{(2k+1)^3} \right) e^{-(2k+1)\pi\mu/\omega_0}.
 \tag{9.18}$$

¹³The integration contour is again that of figure 1, but the poles of the integrand are now at $\pi(2k+1)i$ for $k \in \mathbf{Z}$. Hence, the radii of the two semi-circles can be taken to zero, and only the pole at πi contributes to the integral.

It would be desirable to identify the precise symmetry (perhaps akin to supersymmetry) responsible for the exact vanishing of the vacuum energy to all orders in perturbation theory, but perhaps violated by the nonperturbative effects.

- $|\tilde{M}, \mu\rangle$ should clearly be considered a hydrodynamic state, although it is somewhat outside of the class of hydrodynamic states that solve the equations of motion for the semiclassical Fermi surface. Indeed, due to the staggered manner of how states of different q are filled, the average density of fermions is continuous across the surface of Fermi energy $-\mu$. Perhaps a more useful semiclassical observable would be the staggered density of eigenvalues, \tilde{O} , defined as in (8.3) with an additional insertion of $(-1)^q$ in each sector of angular momentum q .

10. Time-dependent solutions

We can generate some time-dependent solutions by continuing the strategy from the previous section. In particular, we can modify a given Fermi surface by adding conserved quantities that explicitly contain t . This is very similar to the strategy used in noncritical string theory in [55]–[59]. We can immediately write a family of time-dependent solutions, by simply postulating a Fermi surface

$$\sum_{n_i, m_i=0}^{\infty} \tau_{m_1 m_2 n_1 n_2} W_{m_1 m_2 n_1 n_2}(p_i, \lambda_i, t) = 0, \tag{10.1}$$

where $W_{m_1 m_2 n_1 n_2}$ is the basis (8.7) of the \mathcal{W} symmetry algebra, and $\tau_{m_1 m_2 n_1 n_2}$ are arbitrary constants. Note that τ_{0000} effectively plays the role of the scaling variable μ , since it multiplies the central element $W_{0000} \sim 1$ of the symmetry algebra.

The family of static solutions (9.12) is in this class, with only $\tau_{1010} = +\tau_{0101}, \tau_{1001} = -\tau_{0110}$, and τ_{0000} nonzero.

We can now look at some examples of time-dependent solutions from this class.

10.1 Losing or gaining a dimension

The simplest time-dependent solutions are obtained by adding to the Hamiltonian terms linear in a_i and b_i . Of such solutions, the simplest will give the following time-dependent Fermi surface,

$$\frac{1}{2} (p_x^2 + p_y^2 - x^2 - y^2) + c(p_y - y)e^t = -\mu, \tag{10.2}$$

where $c = \tau_{0001}/\sqrt{2}$ is a constant. In the asymptotic past, $t \rightarrow -\infty$, the effect of the time-dependent deformation is negligible, and the Fermi surface approaches the static Fermi surface of the M-theory vacuum in $2 + 1$ dimensions. At late times $t \rightarrow \infty$, however, the Fermi sea is partially drained. Another, similar solution is given by

$$\frac{1}{2} (p_x^2 + p_y^2 - x^2 - y^2) + \frac{\tilde{c}}{2} (p_y - y)^2 e^{2t} = -\mu, \tag{10.3}$$

with $\tilde{c} = \tau_{0002}$ again a constant. This again describes a solution that starts off as the M-theory vacuum, whose Fermi sea is drained at late times everywhere except along the

hypersurface $y = p_y$, where the Fermi sea stays at $-\mu$. Along this hypersurface, the conserved quantity $\nu_y \equiv a_2 b_2$ vanishes. Recalling our construction of the Type 0B string theory vacuum in M-theory, in which only states with $\nu_y = 0$ were filled up to Fermi level $-\mu$, it is natural to identify the time-dependent solution (10.3) as decaying at late times into the Type 0B vacuum. In the process, the effective spacetime dimension changes from $2 + 1$ to $1 + 1$.

Similarly, solutions with τ_{0010} or τ_{0020} nonzero will correspond to the time reversal of (10.2) and (10.3),

$$\frac{1}{2} (p_x^2 + p_y^2 - x^2 - y^2) - c(p_y + y)e^{-t} = -\mu, \tag{10.4}$$

and

$$\frac{1}{2} (p_x^2 + p_y^2 - x^2 - y^2) + \frac{\tilde{c}}{2} (p_y + y)^2 e^{-2t} = -\mu. \tag{10.5}$$

In particular, (10.5) can be interpreted as the time-dependent Fermi surface of a solution that starts off as the Type 0B vacuum at early times, and then evolves into the M-theory vacuum at late times.

10.2 Solutions interpolating between two string vacua

The ingredients of time-dependent solutions from the previous subsection can be easily combined, to construct a solution interpolating between two string theories, via an intermediate M-theory phase. Consider for example

$$\begin{aligned} \frac{1}{2} (p_x^2 + p_y^2 - x^2 - y^2) + \frac{1}{2} [c_1(p_x - x)^2 + c_2(p_y - y)^2] e^{2t} \\ + \frac{1}{2} [c_3(p_x + x)^2 + c_4(p_y + y)^2] e^{-2t} = -\mu. \end{aligned} \tag{10.6}$$

Here c_1, \dots, c_4 are again constants that can be chosen arbitrarily. With only c_1 and c_4 nonzero and positive, (10.6) is the Fermi surface of a time-dependent solution that starts at early times as Type 0B with x playing the role of the spatial dimension, and decays at late times into another Type 0B vacuum, now with y playing the role of the spatial dimension. At times of order $t \approx 0$, this solution is going through a $2 + 1$ dimensional M-theory phase, with the Fermi surface filled more democratically in the x, y plane.

In principle, even though the spacetime dimension may be changing, the free fermion formulation still defines a unitary quantum evolution, and can be used to define an S-matrix between initial and final states, as defined in the asymptotic Type 0B string vacua where they are represented by the massless modes of the Type 0B tachyon. This is a novelty compared to time-dependent solutions found in two-dimensional string theory [55]–[59]: We can now have “decays” of spacetime with well-understood initial *and* final states simultaneously, both being described by a known semiclassical string vacuum. Cosmological decays into “nothing” are also possible, for example with both c_1 and c_2 nonzero.

Clearly, vast families of similar solutions exist, and one can engineer solutions that for example begin in the Type 0A vacuum and evolve into the Type 0B vacuum.

11. Conclusions

In this paper, we have presented a fully nonperturbative definition of noncritical M-theory in $2 + 1$ dimensions, in terms of a double-scaling limit of a nonrelativistic Fermi liquid. Clearly, in our analysis of this theory, we have only scratched the proverbial surface. The exact solvability of the model allows one to extract a wealth of data about this incarnation of M-theory, including detailed information about some of its exotic time-dependent solutions.

The theory is fully defined as a quantum-mechanical theory in terms of the fermions, even in regimes where a semiclassical spacetime interpretation ceases to be valid. In this picture, the fundamental degrees of freedom of M-theory are the elementary fermions of the Fermi liquid.

The fundamental fermions originate in the underlying system of D0- and anti D0-branes of the two-dimensional Type 0A string theory. In this respect, our nonperturbative definition of noncritical M-theory bears striking resemblance to M(atrrix) theory [60]–[62] — another candidate for a nonperturbative formulation of M-theory, defined in terms of the supersymmetric quantum mechanics of N D0-branes of Type IIA string theory in the Sen-Seiberg scaling limit. A possible relation between these two approaches might involve ideas presented in [63].

Our noncritical M-theory provides a unified framework for the dynamics of two-dimensional noncritical strings. Noncritical strings can be embedded into critical string theory via their relation to the topological strings on singular Calabi-Yau manifolds (see, e.g., [64]). Since the latter have been conjecturally related to a topological M-theory in seven dimensions [65], it would be interesting to see whether an embedding of our noncritical M-theory into critical string/M-theory can shed light on the seven-dimensional topological M-theory.

Using the Fermi liquid picture, we have established that noncritical M-theory in $2 + 1$ dimensions can be defined. However, many open questions clearly remain. For example, it is unclear how to formulate this theory directly in terms of a matrix model. An even more pressing challenge is to understand the effective spacetime description of noncritical M-theory vacua, in a language that directly refers to gravity in $2 + 1$ dimensions. Guided by noncritical string theory, we expect that such an effective spacetime gravity description indeed exists. In this description, we expect a propagating degree of freedom — the M-theory analog of the massless tachyon — coupled to a gravitational sector. In noncritical string theory, the relationship between the eigenvalue space and the spacetime Liouville dimension is known to be subtle, involving a nonlocal Laplace-like transform. Finding its analog in noncritical M-theory represents one of the main challenges. It is also natural to ask what is the full spectrum of solitons in the theory, and in particular, whether or not the noncritical M-theory vacuum contains membranes.

In its fermionic formulation, our noncritical M-theory is a rather unique theory, specified by the underlying infinite \mathcal{W} symmetry of its Lagrangian.¹⁴ One natural extension, compatible with the \mathcal{W} symmetry, would be the addition of spin to the fermions. Perhaps this possibility may be related to the existence of two different RR gauge fields in Type

¹⁴This point emerged from discussions with Shamit Kachru.

0A theory. In principle, one can also try to include the nonsinglet states,¹⁵ although it is unclear — in the absence of a direct matrix model formulation — how they can be accommodated and what physical role they will play in the theory.

As to the hope that noncritical M-theory may teach us valuable lessons about the mysterious aspects of M-theory, it is encouraging to see that this theory is described by an exactly solvable system. Exact results for various physical observables are now in principle available, and the challenge is to interpret them and draw the corresponding lessons. The exact evaluation of the vacuum energy in M-theory (essentially, the cosmological constant of the vacuum) performed in this paper is an example, in which several surprising features have been observed.

Our noncritical M-theory may also be of interest from the general viewpoint of quantum gravity. In $2 + 1$ dimensions, there are essentially two successful approaches to quantum gravity, each with its own drawbacks. The first one is the Chern-Simons formulation [66, 67], in which the topological nature of the theory is prominent. However, it is difficult to include any propagating degrees of freedom in this framework. The second possibility is to study compactifications of the full critical string/M theory to $2 + 1$ dimensions, for example on AdS_3 . This also defines a consistent quantum gravitational system, at the cost of carrying the entire baggage of the stringy and KK degrees of freedom. The noncritical M-theory defined in this paper may represent a middle road to quantum gravity in $2 + 1$ dimensions, allowing a propagating degree of freedom but sharing some of the topological features with the Chern-Simons approach.

It is worth pointing out that noncritical M-theory represents a framework in which the physical spacetime is an emergent property, available only for those solutions of the underlying quantum mechanical system that admit a hydrodynamic description. Moreover, this theory seems to be a realization of Mach's principle [68]: The semiclassical physical spacetime is sustained by the collective motion of N fundamental constituent fermions. Without the constituents, there is no hydrodynamics of the Fermi liquid, and consequently no spacetime.

Acknowledgments

We wish to thank Josh Friess, Eric Gimon, Shamit Kachru, Peter Shepard, and Herman Verlinde for useful discussions. The results reported in this paper were presented at TASI on Particle Physics in Boulder in June 2005 (by CAK), and at the Strings 2005 Conference in Toronto in July 2005 (by PH). We would like to thank the organizers of these meetings for their kind hospitality and the opportunity to present our results. This material is based upon work supported by NSF grant PHY-0244900, DOE grant DE-AC02-05CH11231, an NSF Graduate Research Fellowship, and the Berkeley Center for Theoretical Physics.

References

- [1] M.R. Douglas et al., *A new hat for the $c = 1$ matrix model*, [hep-th/0307195](#).

¹⁵We thank Herman Verlinde for raising this issue.

- [2] T. Takayanagi and N. Toumbas, *A matrix model dual of Type 0B string theory in two dimensions*, *JHEP* **07** (2003) 064 [[hep-th/0307083](#)].
- [3] P.H. Ginsparg and G.W. Moore, *Lectures on 2D gravity and 2D string theory*, [hep-th/9304011](#).
- [4] I.R. Klebanov, *String theory in two-dimensions*, [hep-th/9108019](#).
- [5] J. Polchinski, *What is string theory?*, [hep-th/9411028](#).
- [6] S. Alexandrov, *Matrix quantum mechanics and two-dimensional string theory in non-trivial backgrounds*, [hep-th/0311273](#).
- [7] Y. Nakayama, *Liouville field theory: a decade after the revolution*, *Int. J. Mod. Phys. A* **19** (2004) 2771 [[hep-th/0402009](#)].
- [8] E.J. Martinec, *The annular report on non-critical string theory*, [hep-th/0305148](#).
- [9] J. McGreevy and H.L. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *JHEP* **12** (2003) 054 [[hep-th/0304224](#)].
- [10] V.A. Kazakov and A.A. Migdal, *Recent progress in the theory of noncritical strings*, *Nucl. Phys. B* **311** (1988) 171.
- [11] D.J. Gross and N. Miljkovic, *A nonperturbative solution of $D = 1$ string theory*, *Phys. Lett. B* **238** (1990) 217.
- [12] D.J. Gross and I.R. Klebanov, *One-dimensional string theory on a circle*, *Nucl. Phys. B* **344** (1990) 475.
- [13] E. Brézin, V.A. Kazakov and A.B. Zamolodchikov, *Scaling violation in a field theory of closed strings in one physical dimension*, *Nucl. Phys. B* **338** (1990) 673;
P.H. Ginsparg and J. Zinn-Justin, *2-D gravity + 1-D matter*, *Phys. Lett. B* **240** (1990) 333.
- [14] V. Kazakov, *Bosonic strings and string field theories in one-dimensional target space*, LPTENS-90-30, in *Random surfaces and quantum gravity*, Cargèse Proceedings (1990).
- [15] G.W. Moore, *Double scaled field theory at $c = 1$* , *Nucl. Phys. B* **368** (1992) 557.
- [16] A. Jevicki and T. Yoneya, *A deformed matrix model and the black hole background in two-dimensional string theory*, *Nucl. Phys. B* **411** (1994) 64 [[hep-th/9305109](#)].
- [17] U.H. Danielsson, *A matrix model black hole*, *Nucl. Phys. B* **410** (1993) 395 [[hep-th/9306063](#)]; *The deformed matrix model at finite radius and a new duality symmetry*, *Phys. Lett. B* **325** (1994) 33 [[hep-th/9309157](#)]; *Two-dimensional string theory, topological field theories and the deformed matrix model*, *Nucl. Phys. B* **425** (1994) 261 [[hep-th/9401135](#)]; *The scattering of strings in a black hole background*, *Phys. Lett. B* **338** (1994) 158 [[hep-th/9405052](#)]; *A matrix model black hole: act II*, *JHEP* **02** (2004) 067 [[hep-th/0312203](#)];
U.H. Danielsson, N. Johansson, M. Larfors, M.E. Olsson and M. Vonk, *4D black holes and holomorphic factorization of the 0A matrix model*, *JHEP* **10** (2005) 046 [[hep-th/0506219](#)].
- [18] K. Demeterfi and J.P. Rodrigues, *States and quantum effects in the collective field theory of a deformed matrix model*, *Nucl. Phys. B* **415** (1994) 3 [[hep-th/9306141](#)].
- [19] K. Demeterfi, I.R. Klebanov and J.P. Rodrigues, *The exact S matrix of the deformed $c = 1$ matrix model*, *Phys. Rev. Lett.* **71** (1993) 3409 [[hep-th/9308036](#)].

- [20] S. Gukov, T. Takayanagi and N. Toumbas, *Flux backgrounds in 2D string theory*, *JHEP* **03** (2004) 017 [[hep-th/0312208](#)].
- [21] T. Takayanagi, *Comments on 2D Type IIA string and matrix model*, *JHEP* **11** (2004) 030 [[hep-th/0408086](#)].
- [22] N. Seiberg, *Observations on the moduli space of two dimensional string theory*, *JHEP* **03** (2005) 010 [[hep-th/0502156](#)].
- [23] O. Bergman and M.R. Gaberdiel, *Dualities of Type 0 strings*, *JHEP* **07** (1999) 022 [[hep-th/9906055](#)].
- [24] P. Hořava, *Stability of Fermi surfaces and K-theory*, *Phys. Rev. Lett.* **95** (2005) 016404 [[hep-th/0503006](#)].
- [25] L.G. Yaffe, *Large- N limits as classical mechanics*, *Rev. Mod. Phys.* **54** (1982) 407.
- [26] P. Hořava and E. Witten, *Heterotic and Type-I string dynamics from eleven dimensions*, *Nucl. Phys. B* **460** (1996) 506 [[hep-th/9510209](#)]; *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys. B* **475** (1996) 94 [[hep-th/9603142](#)].
- [27] D. Kutasov and N. Seiberg, *Noncritical superstrings*, *Phys. Lett. B* **251** (1990) 67; S. Murthy, *Notes on non-critical superstrings in various dimensions*, *JHEP* **11** (2003) 056 [[hep-th/0305197](#)].
- [28] P. Hořava, *M-theory as a holographic field theory*, *Phys. Rev. D* **59** (1999) 046004 [[hep-th/9712130](#)].
- [29] P. Hořava and D. Minic, *Probable values of the cosmological constant in a holographic theory*, *Phys. Rev. Lett.* **85** (2000) 1610 [[hep-th/0001145](#)].
- [30] P. Debye, *Zur Theorie der Spezifischen Wärme*, *Annalen der Physik* **39** (1912) 789.
- [31] See, e.g., section 5 of M. Le Bellac, F. Mortessagne and G.G. Batrouni, *Equilibrium and non-equilibrium statistical thermodynamics*, Cambridge University Press, Cambridge U.K. (2004).
- [32] A. Okounkov, N. Reshetikhin and C. Vafa, *Quantum Calabi-Yau and classical crystals*, [hep-th/0309208](#).
- [33] S.R. Das and A. Jevicki, *String field theory and physical interpretation of $D = 1$ strings*, *Mod. Phys. Lett. A* **5** (1990) 1639.
- [34] F.D.M. Haldane, in *Perspectives in many-particle physics*, eds. R.A. Broglia and J.R. Schrieffer, North Holland (1994).
- [35] A. Houghton and J.B. Marston, *Bosonization and fermion liquids in dimensions greater than one*, [cond-mat/9210007](#);
A. Houghton, H.-J. Kwon and J.B. Marston, *Multidimensional bosonization*, [cond-mat/9810388](#);
A.H. Castro Neto and E. Fradkin, *Bosonization of the low energy excitations of Fermi liquids*, [cond-mat/9304014](#); *Bosonization of Fermi liquids*, [cond-mat/9307005](#); *Exact solution of the Landau fixed point via bosonization*, [cond-mat/9310046](#).
- [36] R. Shankar, *Renormalization group approach to interacting fermions*, *Rev. Mod. Phys.* **66** (1994) 129.

- [37] G.W. Moore, M.R. Plesser and S. Ramgoolam, *Exact S matrix for 2D string theory*, *Nucl. Phys. B* **377** (1992) 143 [[hep-th/9111035](#)].
- [38] O. DeWolfe, R. Roiban, M. Spradlin, A. Volovich and J. Walcher, *On the S-matrix of Type 0 string theory*, *JHEP* **11** (2003) 012 [[hep-th/0309148](#)].
- [39] E. Witten, *Ground ring of two-dimensional string theory*, *Nucl. Phys. B* **373** (1992) 187 [[hep-th/9108004](#)]; E. Witten and B. Zwiebach, *Algebraic structures and differential geometry in 2D string theory*, *Nucl. Phys. B* **377** (1992) 55 [[hep-th/9201056](#)].
- [40] J. Polchinski, *Critical behavior of random surfaces in one-dimension*, *Nucl. Phys. B* **346** (1990) 253; *Classical limit of (1+1)-dimensional string theory*, *Nucl. Phys. B* **362** (1991) 125.
- [41] D. Minic, J. Polchinski and Z. Yang, *Translation invariant backgrounds in (1+1)-dimensional string theory*, *Nucl. Phys. B* **369** (1992) 324.
- [42] A. Kapustin, *Noncritical superstrings in a Ramond-Ramond background*, *JHEP* **06** (2004) 024 [[hep-th/0308119](#)].
- [43] H.L. Verlinde, *Superstrings on AdS_2 and superconformal matrix quantum mechanics*, [hep-th/0403024](#).
- [44] A. Strominger, *A matrix model for AdS_2* , *JHEP* **03** (2004) 066 [[hep-th/0312194](#)].
- [45] J. Schwinger, *Brownian motion of a quantum oscillator*, *J. Math. Phys.* **2** (1961) 407; L.V. Keldysh, *Diagram technique for nonequilibrium processes*, *Zh. Eksp. Teor. Fiz.* **47** (1964) 1515 [*Sov. Phys. JETP* **20** (1964) 1018].
- [46] A.J. Niemi and G.W. Semenoff, *Finite temperature quantum field theory in Minkowski space*, *Ann. Phys. (NY)* **152** (1984) 105; *Thermodynamic calculations in relativistic finite-temperature quantum field theories*, *Nucl. Phys. B* **230** (1984) 181.
- [47] H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States*, North Holland, Amsterdam (1982).
- [48] J.M. Maldacena, *Eternal black holes in Anti-de-Sitter*, *JHEP* **04** (2003) 021 [[hep-th/0106112](#)].
- [49] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, *The black hole singularity in AdS/CFT* , *JHEP* **02** (2004) 014 [[hep-th/0306170](#)].
- [50] N. Itzhaki and J. McGreevy, *The large- N harmonic oscillator as a string theory*, *Phys. Rev. D* **71** (2005) 025003 [[hep-th/0408180](#)].
- [51] A. Boyarsky, V.V. Cheianov and O. Ruchayskiy, *Fermions in the harmonic potential and string theory*, *JHEP* **01** (2005) 010 [[hep-th/0409129](#)].
- [52] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [[hep-th/0111222](#)].
- [53] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [[hep-th/0403110](#)].
- [54] J.M. Maldacena and N. Seiberg, *Flux-vacua in two dimensional string theory*, *JHEP* **09** (2005) 077 [[hep-th/0506141](#)].
- [55] J.L. Karczmarek and A. Strominger, *Matrix cosmology*, *JHEP* **04** (2004) 055 [[hep-th/0309138](#)]; *Closed string tachyon condensation at $c = 1$* , *JHEP* **05** (2004) 062 [[hep-th/0403169](#)].

- [56] S.R. Das, J.L. Davis, F. Larsen and P. Mukhopadhyay, *Particle production in matrix cosmology*, *Phys. Rev. D* **70** (2004) 044017 [[hep-th/0403275](#)].
- [57] J.L. Karczmarek, A. Maloney and A. Strominger, *Hartle-Hawking vacuum for $c = 1$ tachyon condensation*, *JHEP* **12** (2004) 027 [[hep-th/0405092](#)].
- [58] S.R. Das and J.L. Karczmarek, *Spacelike boundaries from the $c = 1$ matrix model*, *Phys. Rev. D* **71** (2005) 086006 [[hep-th/0412093](#)].
- [59] S.R. Das, *D-branes in 2D string theory and classical limits*, [hep-th/0401067](#); *Non-trivial 2D space-times from matrices*, [hep-th/0503002](#).
- [60] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev. D* **55** (1997) 5112 [[hep-th/9610043](#)].
- [61] A. Sen, *D0 branes on T^n and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 51 [[hep-th/9709220](#)]. N. Seiberg, *Why is the matrix model correct?*, *Phys. Rev. Lett.* **79** (1997) 3577 [[hep-th/9710009](#)].
- [62] J. Polchinski, *M-theory and the light cone*, *Prog. Theor. Phys. Suppl.* **134** (1999) 158 [[hep-th/9903165](#)].
- [63] P. Hořava, *Type IIA D-branes, K-theory and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1999) 1373 [[hep-th/9812135](#)].
- [64] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño and C. Vafa, *Topological strings and integrable hierarchies*, *Commun. Math. Phys.* **261** (2006) 451 [[hep-th/0312085](#)].
- [65] R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa, *Topological M-theory as unification of form theories of gravity*, *Adv. Theor. Math. Phys.* **9** (2005) 603 [[hep-th/0411073](#)].
- [66] E. Witten, *(2 + 1)-dimensional gravity as an exactly soluble system*, *Nucl. Phys. B* **311** (1988) 46; *Topology changing amplitudes in (2 + 1)-dimensional gravity*, *Nucl. Phys. B* **323** (1989) 113.
- [67] S. Carlip, *Quantum gravity in 2 + 1 dimensions*, Cambridge University Press, Cambridge U.K. (1998).
- [68] E. Mach, *Die Mechanik in ihrer Entwicklung*, Brockhaus, Leipzig (1883).